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An Application of Topological Dynamics to Obtain a New Invariance Property for Nonautonomous Ordinary Differential Equations

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1. INTRODUCTION

Peng considered the differential equations:

$$\dot{x}(t) = f(x(t), q(t)), \qquad x(0) = x_0, \qquad q(t) \in Q \tag{1}$$

where $x(t) \in R^n$, real n-dimensional space, $t \in R$, and Q is a compact set in R^m. He assumed that f and $\partial f/\partial x$ are continuous in (x, q). A measurable function q_0 from R to Q is called *admissible*, and for a particular x_0 and admissible function q_0, the corresponding trajectory is $\phi(x_0, q_0, t)$; that is, $\phi(x_0, q_0, t)$ is the solution of $\dot{x} = f(x(t), q(t))$ satisfying $x(0) = x_0$.

Peng defines a positively quasi-invariant set N as a set such that for any x_0 in N, there is a sequence $\{q^k\}$ of admissible functions such that as $k \to \infty$, $\phi(x_0, q^k, t)$ converges to a function $\psi(t)$ uniformly on bounded sets in $(0, \infty)$ and $\psi(t)$ remains in N for all $t > 0$.

He then established under an assumption A1 that if $\phi(x_0, q, t)$ is bounded as $t \to \infty$ for some admissible q, then the positive limit set $P^+(x_0, q)$ is a positively quasi-invariant set. Assumption A1 is that for each $T \geqslant 0$ and each bounded set B in R^n, there exists a bounded set $A(B, T)$ such that $\phi(x_0, q, t)$ is in $A(B, T)$ for all admissible q's, all x_0 in B and all t in $[0, T]$.

This result implies that if there exists some set E in R^n such that solutions $\phi(x_0, q, t)$ for each admissible q approach E as $t \to \infty$, then all bounded solutions must approach the largest positively quasi-invariant subset of E as $t \to \infty$. This invariance principle follows by observing that (1) each bounded solution approaches its positive limit set as $t \to \infty$, (2) the positive limit sets are each positively quasi invariant and are contained in E, and (3) the union of positively quasi-invariant sets is positively quasi invariant.

In a typical application, Liapunov theory may be used to show that all solutions are bounded and approach a subset of R^n as $t \to \infty$. It may be possible to show that the origin is the only positively quasi-invariant set contained in this subset. Then it follows that all solutions approach the origin as $t \to \infty$. A generalized version of this invariance principle is established in Section 7 of this paper.

Peng uses his invariance principle to obtain conditions for a Liapunov function $V(x)$ to be an upper bound for the cost functional of a feedback control problem when the unknown parameter $q(t)$ may be any admissible function. He also obtains new sufficient conditions for the asymptotic stability of nonautonomous systems.

The theory of dynamical systems has often been used in studying invariance and stability properties for autonomous, periodic, and almost periodic differential equations [7–9]. More recently Miller and Sell [2–6] have shown how this approach, using a theory of local semiflows, can be extended to include a class of nonautonomous differential equations and Volterra integral equations. They define a local semiflow function π from $Y \times W \times R^+$ into $Y \times W$ where W is an open subset of a metric space $\underline{X}$, R^+ is $(0, \infty)$, and Y is a metric space on a set of functions from $W \times R^+$ into R^n. For a point p in $Y \times W$, $p = [f, x_0]$;

$$\pi(p, t) \equiv [f_t, \phi(x_0, f, t)] \tag{2}$$

where $f_t(x, \tau) \equiv f(x, \tau + t)$ for all (x, τ) in the domain of f and $\phi(x_0, f, t)$ is the solution of

$$x(t) = x_0 + \int_0^t f(x(s), s)\, ds.$$

The space $\underline{X}$ is R^n for differential equations and is an appropriate space of functions in applications to Volterra integral equations.

The results of Miller and Sell are of considerable generality and one would expect that they contained Peng's invariance property as a special case. It comes, therefore, as a surprise to discover that the differential equations considered by Peng do not satisfy the conditions assumed by Miller and Sell for an invariance property. The difficulty is that they assume that the motion $\pi(p, t)$ as t varies from 0 to ∞ for a fixed p must be contained in a compact set in $Y \times W$. In order for this to be true, not only must the solution to the differential equation be contained in a compact set in W, but the closure in Y of the set of functions $\{f_t: t \in [0, \infty)\}$ must be compact. It is easy to produce a function which satisfies Peng's invariance property but which does not satisfy the above compactness criteria for any of the spaces Y considered by Miller and Sell.

In this paper a different metric space is used and verifiable conditions on f are obtained which assure that the desired sufficient compactness criteria are satisfied. Peng's results are then extended using this metric.

In Section 2 of this paper basic results of the Miller and Sell theory are summarized and it is shown that they do not cover Peng's invariance principle. Section 3 describes a different topology on a space of functions which comprise the right-hand side of an ordinary differential equation. Section 4 gives conditions on the function f sufficient to assure that the function π defined in (2) generates a local semiflow with this topology. In Section 5 it is shown that under a convenient set of verifiable conditions the motion of the function π is contained in a compact set in the range space. Section 6 generalizes some of Sell's results and applies them to prove and extend Peng's invariance result. Section 7 presents a generalized invariance principle along with applications using Liapunov stability theory. Section 8 suggests the use of a similar topology for functional differential equations.

2. Background and Definitions

In Refs. [2, 3] Sell considers differential equations of the form $\dot{x} = f(x, t)$ for continuous functions f from $W \times R \to R^n$ where W is an open set in R^n. He uses the function π in (2) above to construct what he calls a local dynamical system on $\mathscr{F} \times W$ where $\mathscr{F}$ is the closure under the metric ρ_c (defined below) of the set $\{f_t: t \in (-\infty, \infty)\}$. He then defines limiting equations as differential equations in which the function is the first component of a point in the positive limit set for the motion of the local dynamical system. Several invariance and stability results are obtained for solutions of the limiting equations and the behavior of solutions of the original equation is related to the behavior of solutions to these limiting equations. Some of these results are quite similar in form to those of Peng.

In order to assure the existence of limiting equations as well as to prove the invariance theorems, the trajectory $\pi(p, t)$ for $t \geqslant 0$ must be in a compact set in $\mathscr{F} \times W$. It is in this requirement that the topology on $\mathscr{F}$ becomes crucial. Sell's metric ρ_c is defined for $f, g \in \mathscr{F}$ as follows. Let $\{K_n\}_{n=1}^{\infty}$ be a sequence of compact sets in $W \times R$ such that $K_n \subset K_{n+1}$, $n = 1, 2,\ldots$, and

$$\bigcup_{n=1}^{\infty} K_n = W \times R.$$

Then

$$\rho_c(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\sup\{|f(x, t) - g(x, t)|: (x, t) \in K_n\}}{1 + \sup\{|f(x, t) - g(x, t)|: (x, t) \in K_n\}}. \tag{3}$$

The resulting topology can also be characterized by its convergence properties. If $\{f_n\}_{n=1}^{\infty}$ is a sequence in $\mathscr{F}$, then $f_n \to f_0$ in $\mathscr{F}$ as $n \to \infty$ if and only if

$f_n(x, t) \to f_0(x, t)$ as $n \to \infty$ for each (x, t) in $W \times R$ and the convergence is uniform on compact subsets of $W \times R$.

Miller and Sell [4, 5] have extended the above approach to study Volterra integral equations of the form

$$x(t) = f(t) + \int_0^t g[t, s, x(s)]\, ds \tag{4}$$

for appropriate classes of functions f and g having range in R^n. They use another topology denoted by $G_p{}^*$ for the t translates of g. The topology $G_p{}^*$ (defined subsequently) can be applied to functions of two variables (t and x) and turns out to be weaker than $\mathscr{F}$.

Miller [6] recognized the fact that one is usually interested in the positive motion of a system and he defined a local semiflow as a formalization of Sell's local dynamical system restricted to $t \geqslant 0$. His definition which will be used for this paper is given below.

DEFINITION 1. A local semiflow on a metric space $\underline{X}$ is defined by Miller [6] as follows. Let $S = \{(p, t): p \in \underline{X}, t \in [0, \alpha(p))$ where $0 < \alpha(p) \leqslant \infty\}$ and let π be a function from S into $\underline{X}$. Then the pair (π, S) is called a local semiflow on $\underline{X}$ if the following conditions are satisfied:

(S1) π is continuous on S in the pair (p, t);

(S2) $\pi(p, 0) = p$ for all p in $\underline{X}$;

(S3) if $t \in [0, \alpha(p))$ and $r \in [0, \alpha(\pi(p, t)))$, then $t + r \in [0, \alpha(p))$ and $\pi[\pi(p, t), r] = \pi(p, t + r)$;

(S4) for any sequence p_n in $\underline{X}$ if $p_n \to p$, then $\alpha(p) \leqslant \liminf \alpha(p_n)$ as $n \to \infty$;

(S5) the intervals $[0, \alpha(p))$ are maximal in the sense that if $\alpha(p) < \infty$ then the closure of the set $\{\pi(p, t); t \in [0, \alpha(p))\}$ is not compact in $\underline{X}$.

The orbit through p is the set $\gamma(p) = \{\pi(p, t); t \in [0, \alpha(p))\}$ and for points p such that $\alpha(p) = \infty$, the ω-limit set $\Omega(p)$ is defined by $\Omega(p) = \bigcap \{\text{closure } \gamma[\pi(p, t)]: t \in [0, \infty)\}$. A subset A of $\underline{X}$ is called positively invariant with respect to π if $p \in A$ implies $\alpha(p) = \infty$ and $\gamma(p) \subset A$.

Sell [2] defined a local dynamical system in essentially the same fashion except that π is also assumed to have a maximal negative extension. If $\pi(p, t)$ is defined on $(-\infty, \infty)$ for each $p \in A$ and remains in A for all $t \in (-\infty, \infty)$, then A is called invariant with respect to π. A standard result in Ref. [10] for general dynamical systems is that if $\gamma(p)$ is contained in a compact subset of $\underline{X}$, then $\alpha(p) = \infty$ and $\Omega(p)$ is nonempty, compact, and invariant with respect to π.

In this paper the following type of nonautonomous differential equation is considered

$$\dot{x} = f(x, t), \qquad x(0) = x_0\,, \tag{5}$$

or in integral form

$$x(t) = x_0 + \int_0^t f(x(s), s)\, ds.$$

It is assumed that f is defined on $W \times R^+$ (W is an open set in R^n), is continuous in x for each t and measurable in t for each x and satisfies the following conditions for each compact set $A \subset W$

$$|f(x, t)| \leqslant m_A(t) \text{ for all } x \in A \text{ where } m_A(t) \text{ is locally } L_1(R^+; R^+), \text{ and} \tag{T1}$$

$$|f(x, t) - f(y, t)| \leqslant K_A(t)\,|x - y| \text{ for all } x, y \in A \text{ and } K_A(t) \text{ is locally } L_1(R^+; R^+). \tag{T2}$$

A function m is called locally L_p if for each bounded measurable set B in R^+, $\int_B |m(s)|^p\, ds$ is bounded.

Miller and Sell [4] have shown that if f satisfies (T1) and (T2), then for each $x_0 \in W$, (5) has a unique solution $\phi(x_0, f, t)$ in W defined on a nonempty maximal interval of definition and satisfying the differential equation for t almost everywhere in R^+.

The semiflow function π defined in Eq. (2) is used by Miller and Sell [4, 5] for functions satisfying (T1) and (T2). The metric on which the semiflow is defined is the cross-product of a specified function space (for example $G_p{}^*$) and W.

Properties (S2), (S3), and (S5) follow immediately from Ref. [4] and the definition of π (with $[0, \alpha(f, x))$ the maximal interval of definition of the solution $\phi(x, f, t)$ on R^+). In order for (S1) and (S4) to be meaningful the functions f_t must be imbedded in an appropriate metric topology. Under the topology $\mathscr{F}$ used by Sell with the metric ρ_c, the space of continuous functions from $W \times R \to R^n$ is a complete metric space [2]. The topology $G_p{}^*$ is used by Miller and Sell [4, 5] for functions satisfying (T1) and (T2) with the functions m_A and K_A locally $L_p(R^+; R^+)$. For some $p \in [1, \infty)$, $f_n \to f$ in $G_p{}^*$ as $n \to \infty$ if for each compact set K in W and each compact set I in R^+,

$$\sup \left[\int_I |f_n(x(s), s) - f(x(s), s)|^p\, ds \colon x \in C(I; K)\right] \to 0 \qquad \text{as} \quad n \to \infty. \tag{6}$$

The set $C(I, K)$ is the set of continuous functions from I to K. This topology is also metrizable [6] and it is easily shown that $\mathscr{F}$ is a stronger topology than $G_p{}^*$.

Using the topology $\mathscr{F}$ and the theory of topological dynamics, Sell [3] obtains invariance results such as the following: if f is a continuous function from $W \times R \to R^n$ and if the associated differential equation has unique solutions, then if the motion $\pi(f, x, t)$ for $t \in [0, \infty)$ is contained in a compact set in $\mathscr{F} \times W$, and if (f^*, x^*) is in its positive limit set, then the solution $\phi(x^*, f^*, t)$ of the associated differential equation is contained for all $t \in R$ in the positive limit set of the solution $\phi(x, f, t)$.

This result is very similar to Peng's result with $\phi(x^*, f^*, t)$ serving the role of Peng's limit function $\psi(t)$, the positive limit set of $\phi(x, f, t)$ serving as the positively quasi-invariant set, and a set f_{t_k} of time translates of f replacing the functions $f(x, q_k(t))$. Peng's result, unlike Sell's, applies to functions which are not necessarily continuous in f. This restriction can be overcome using G_p^*. However, Peng's invariance property cannot be obtained using $\mathscr{F}$ or G_p^* because the set of time translates of some of the functions considered by Peng may not be contained in a compact set in $\mathscr{F}$ or G_p^*. One such function is

$$f(x, q(t)) = q(t) = \sin e^t. \tag{7}$$

It is easily verified that this function satisfies Peng's assumption (A1). If $\{f_t : t \in [0, \infty)\}$ were contained in a compact set in G_p^*, then any sequence $\{f_{t_n}\}_{n=1}^{\infty}$ would have a Cauchy subsequence. Therefore, to prove $\{f_t ; t \in [0, \infty)\}$ is not in a compact set in G_p^* it is sufficient to show that $\{f_{t_n}\}$ is not Cauchy in G_p^* for any sequence with $t_n \to \infty$ as $n \to \infty$. It will then follow that no such subsequence can be Cauchy in the stronger topology $\mathscr{F}$ and, therefore, $\{f_t : t \in [0, \infty)\}$ is not contained in a compact set in $\mathscr{F}$ either.

If $t_n \to \infty$ as $n \to \infty$ and $\{f_{t_n}\}$ were Cauchy in G_p^*, then for each T,

$$\int_0^T |\sin e^{(t+\tau_n)} - \sin e^{(t+\tau_m)}|^p \, dt \to 0 \qquad \text{as} \quad m, n \to \infty. \tag{8}$$

Let $\sigma = e^t$. Then the above integral becomes:

$$\int_1^{e^T} |\sin e^{\tau_n}\sigma - \sin e^{\tau_m}\sigma|^p \frac{d\sigma}{\sigma} \geqslant e^{-T} \int_1^{e^T} |\sin e^{\tau_n}\sigma - \sin e^{\tau_m}\sigma|^p \, d\sigma. \tag{9}$$

First choose τ_n large enough so that $\sin e^{\tau_n}\sigma$ goes through several cycles as σ varies from 1 to e^T. Then if τ_m is chosen such that $e^{\tau_m} \geqslant 3e^{\tau_n}$, then $\sin e^{\tau_m}\sigma$ is negative at least one-third of the time on each interval where $\sin e^{\tau_n}\sigma \geqslant \frac{1}{2}$. Since the latter condition is satisfied one-third of the time on each cycle, the condition on τ_n assures that $\sin e^{\tau_n}\sigma \geqslant \frac{1}{2}$ at least 20% of the time on $[1, e^T]$.

Combining these observations gives:

$$e^{-T} \int_1^{e^T} | \sin e^{\tau_n}\sigma - \sin e^{\tau_m}\sigma |^p \, d\sigma \geqslant (1/15)(\tfrac{1}{2})^p (1 - e^{-T}). \tag{10}$$

This contradiction establishes that $\{f_t \colon t \in R^+\}$ is not compact in $\mathscr{F}$ or $G_p{}^*$.

3. The Topologies $\mathscr{I}$ and $\mathscr{I}_f$

An indefinite integral topology denoted by $\mathscr{I}$ will now be introduced which is weaker than $G_p{}^*$ and which has for a class of functions including those considered by Peng, the property that the set $\{f_t \colon t \in R^+\}$ is contained in a compact subset of $\mathscr{I}$ and properties (S1)–(S5) are satisfied with π as previously defined.

For functions f satisfying (T1) and (T2), clearly all translate functions f_t also satisfy these conditions. For functions satisfying (T1) and (T2) let $f_n \to g$ in $\mathscr{I}$ as $n \to \infty$ mean that for each compact set A in W and each finite T in R^+.

$$\sup_{t \in [0,T], x \in A} \left| \int_0^t [f_n(x, s) - g(x, s)] \, ds \right| \to 0 \qquad \text{as} \quad n \to \infty. \tag{11}$$

This space is metrizable with metric ρ defined by $\rho(f, g) = \rho_c(F, G)$ where ρ_c is the metric for the compact open topology used by Sell [2, 3] and F and G are defined by

$$F(x, t) \equiv \int_0^t f(x, s) \, ds, \qquad G(x, t) \equiv \int_0^t g(x, s) \, ds. \tag{12}$$

Note that $\rho(F, G) = 0$ if and only if for each x, $f(x, t) = g(x, t)$ almost everywhere (a.e.) in t. It will now be shown that this implies as shown below that $\phi(x, f, t) = \phi(x, g, t)$ for all $x \in W$, $t \in [0, \alpha(x))$, where $\alpha(x)$ is the maximal interval on which $\phi(x, f, t)$ (or $\phi(x, g, t)$) is defined.

Suppose that f_1 and f_2 are two functions which each satisfy (T1) and (T2) with bounds and Lipschitz coefficients $m_A{}^1(t)$, $m_A{}^2(t)$, $K_A{}^1(t)$ and $K_A{}^2(t)$ respectively for each compact set A and that for each x, f_1 and f_2 are equal a.e. in t. Then since solutions are unique, to establish that the solutions resulting from (5) are the same, it is sufficient to show that for any continuous function $x(s)$ mapping $R^+ \to W$,

$$\int_0^t [f_1(x(s), s) - f_2(x(s), s)] \, ds \qquad \text{is zero for all} \quad t > 0. \tag{13}$$

Given any $t > 0$ and $\epsilon > 0$, let

$$\eta \equiv \int_0^t (K_A{}^1(s) + K_A{}^2(s)) \, ds$$

for $A = \{x(s); s \in [0, t]\}$ which is clearly a compact set in W. Now choose a piecewise continuous function

$$y(s) = \sum_{i=1}^{M} \chi_i(s) \tag{14}$$

where $t_1 = 0 < t_2 < \cdots < t_M = t$, $\chi_i(s) = y_i$ on $[t_i, t_{i+1}) = 0$ elsewhere, each y_i is in A, and on $[t_i, t_{i+1}]$, $|y_i - x(s)| < \epsilon/\eta$. Such a function exists for some M since x is a continuous function. It then follows quite easily that

$$\begin{aligned}
&\left| \int_0^t [f_1(x(s), s) - f_2(x(s), s)]\, ds \right| \\
&\quad \leqslant \sum_{i=1}^{M} \int_{t_i}^{t_{i+1}} |[f_1(x(s), s) - f_1(y_i, s)]|\, ds + \sum_{i=1}^{M} \int_{t_i}^{t_{i+1}} |f_1(y_i, s) - f_2(y_i, s)| ds \\
&\quad + \sum_{i=1}^{M} \int_{t_i}^{t_{i+1}} |f_2(y_i, s) - f_2(x(s), s)|\, ds < \int_0^t \frac{\epsilon}{\eta} [K_1(s) + K_2(s)]\, ds = \epsilon.
\end{aligned}$$

Since ϵ is arbitrary, the result follows.

That $\mathscr{I}$ is a weaker topology than $G_p{}^*$ can be seen by the following argument. Let A be a compact set in W and $T \in (0, \infty)$. Then for $f, f_n \in \mathscr{I}$.

$$\begin{aligned}
&\sup_{x \in A, t \in [0,T]} \left| \int_0^t [f_n(x, s) - f(x, s)]\, ds \right| \\
&\quad \leqslant \sup_{x \in A, t \in [0,T]} \int_0^t |f_n(x, s) - f(x, s)|\, ds \leqslant \text{(by the Hölder inequality)} \\
&\quad \times \sup_{x \in A} T^{1/q} \left(\int_0^T |f_n(x, s) - f(x, s)|^p\, ds \right)^{1/p} \\
&\quad \leqslant T^{1/q} \sup_{x(\cdot) \in \mathscr{C}([0,T], A)} \left(\int_0^T |f_n(x(s), s) - f(x(s), s)|^p\, ds \right)^{1/p}
\end{aligned}$$

where $(1/p) + (1/q) = 1$ for $p > 1$. For $p = 1$ the next to last step is not needed. $\mathscr{C}([0, T]; A)$ is the set of continuous functions from $[0, T]$ into A. The functions $x(s) = x \in A$ are in this set which gives the last inequality. It follows that if $f = \lim_{n \to \infty} f_n$ in $G_p{}^*$ then f is also the limit in $\mathscr{I}$ so $\mathscr{I}$ is weaker than $G_p{}^*$. That $\mathscr{I}$ is strictly weaker can be seen by considering the sequence $f_k(t) = \sin kt$ which converges to the zero function in $\mathscr{I}$ but does not converge in $G_p{}^*$.

An unfortunate property of $\mathscr{I}$ as presently defined is that it is not a complete metric space. There are sequences $\{f_n\}$ in $\mathscr{I}$ that are Cauchy convergent but cannot have a limit that is continuous in x for every t. For example, suppose $W = (-\infty, \frac{1}{2})$. Define $f(x, t)$: $W \times R^{+} \to R$ by

$$
\begin{aligned}
f(x, t) &= 0 \qquad \text{for} \quad x \leqslant 0 \\
&= 1 - e^{-n(t-n-\frac{1}{2}+x)} \qquad \text{for} \quad 0 < x < \tfrac{1}{2}, \quad \text{and} \\
&\qquad\qquad n + \tfrac{1}{2} - x < t \leqslant n + \tfrac{1}{2}, \quad n = 1, 2, \ldots \\
&= 1 - e^{-n(-t+n+\frac{1}{2}+x)} \qquad \text{for} \quad 0 < x < \tfrac{1}{2}, \quad \text{and} \\
&\qquad\qquad n + \tfrac{1}{2} < t \leqslant n + \tfrac{1}{2} + x \\
&= 0 \qquad \text{elsewhere.}
\end{aligned}
\tag{17}
$$

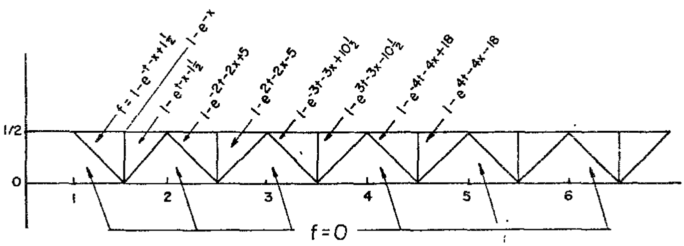

FIGURE 1

(f is illustrated in Fig. 1.) Then $|f| \leqslant 1$ everywhere and f is continuous in x since as $x \to 0$, $|t - n - \frac{1}{2} \pm x| \to 0$ in the interval where f is nonzero so $f(x, t) \to 0$. Also, as $x \to \pm t - n - \frac{1}{2}$ the exponent $\to 0$ and so does f. Also for $n < t \leqslant n + \frac{1}{2}$

$$
\begin{aligned}
|f(x, t) - f(y, t)| &\leqslant |e^{-n(t-n-\frac{1}{2}+x)} - e^{-n(t-n-\frac{1}{2}+y)}| \\
&\leqslant e^{n/2} |e^{-nx} - e^{-ny}| \\
&\leqslant n e^{n/2} |x - y| = K(t) |x - y|,
\end{aligned}
\tag{18}
$$

where $K(t) = n e^{n/2}$ on $n < t \leqslant n + \frac{1}{2}$. A similar result holds for $n + \frac{1}{2} < t \leqslant n + 1$. Therefore, f satisfies (T1) and (T2).

However, $f(x, t + m) \to g$ in $\mathscr{I}$ as $m \to \infty$ where

$$
g = 1 \qquad \text{for} \quad n + \tfrac{1}{2} - x < t < n + \tfrac{1}{2} + x \quad (0 < x < \tfrac{1}{2}), \tag{19}
$$

$n = 0, 1, 2, \ldots$ and $= 0$ elsewhere as shown in Fig. (2).

This is clear since f is uniformly bounded and $\{f_m\}$ converges to g pointwise. But g is discontinuous in x at points where $0 < x < \frac{1}{2}$ and $t = n + \frac{1}{2} \pm x$. It can also be shown that no change in g for each x on a set in t of Lebesgue measure zero will remove these discontinuities.

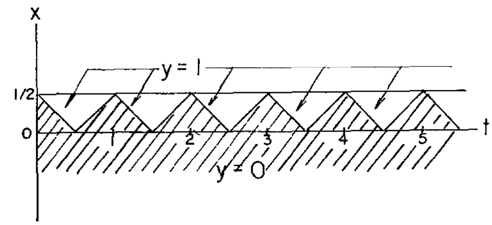

FIGURE 2

The above example is of interest because it illustrates one of the difficulties in establishing that the motion $\pi(f, x, t)$ as previously defined is compact in $\mathscr{I} \times W$. There is also a difficulty in establishing property (S1) (continuity) for π as the following example shows.

Consider the function f: $R \times R^+ \to R$ defined by

$$\begin{aligned} f(x, t) &= 1 + n(t - n - x) \qquad \text{on} \qquad n + x - (1/n) \leqslant t \leqslant n + x, \\ &= 1 - n(t - n - x) \qquad \text{on} \qquad n + x \leqslant t \leqslant n + x + (1/n), \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n = 2, 3, \ldots \\ &= 0 \qquad \text{elsewhere.} \end{aligned} \tag{20}$$

Thus f, as shown in Fig. 3, is a function which has value 1 along the 45° lines passing through $x = 0$, $t = 2, 3, \ldots$, and for each fixed x, f is zero except on successive intervals of width $2/n$ which occurs one unit apart in t. It is easy to see that (T1) and (T2) are satisfied even though the Lipschitz bounds $(K_A(t))$ approach ∞ as $t \to \infty$. The nonzero intervals get narrower as t

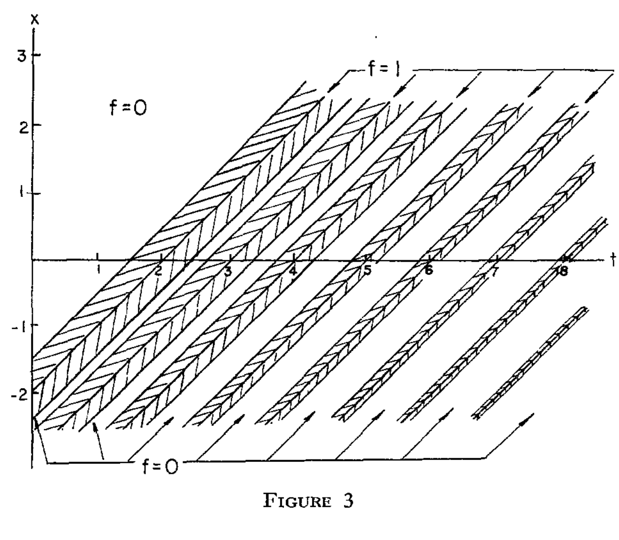

FIGURE 3

increases and f is uniformly bounded, so it is apparent that as t increases, the translate functions f_t approach the zero function in $\mathscr{I}$. In particular, the sequence of translates $\{f_n\}$ converges in $\mathscr{I}$ to the zero function. However, $\phi(0, f_n, t) = t$ for all n. Therefore, $|\phi(0, f_n, t) - \phi(0, 0, t)|$ cannot approach zero as $n \to \infty$ even though $\rho(f_n, 0) \to 0$ as $n \to \infty$. Therefore, π is not continuous on $W \times \mathscr{I}$.

In order to obtain a semiflow with the function π and the metric ρ, the domain of π must be restricted to an appropriate subset of $\mathscr{I}$. Proceeding with the approach used by Sell and Miller, the closed hull of f denoted by $\mathscr{I}_f$ is defined as follows: $g \in \mathscr{I}$ is in $\mathscr{I}_f$ if and only if there exists a sequence $\{t_n\}$ in R^+ such that $\rho(f_{t_n}, g) \to 0$ as $n \to \infty$. Thus $\mathscr{I}_f$ is the closure under ρ of the set of positive translates of f. The above example shows that there are functions in $\mathscr{I}$ such that π is not continuous on $\mathscr{I}_f \times W \times R^+$. The following section gives additional conditions on f which imply that π is a local semiflow on $\mathscr{I}_f \times W$.

Others besides Miller and Sell have considered the use of integral topologies on a class of functions as a means to obtain results on stability and continuous dependence of solution on parameters. Part of the problem of establishing that the function π is continuous on $\mathscr{I}_f \times W \times R^+$ consists in establishing the continuity of solutions with respect to the topology on $\mathscr{I}_f$. There is a considerable body of literature concerned with continuous dependence of solution of parameters. Some references are given at the end of this paper. Others are mentioned in these references.

Kurzweil [14] defines a class $f_{(1)}(G, K_1)$ of continuous functions which satisfy:

$$\|f(x, \tau)\| \leqslant K_1,$$

$$\|f(x_1, \tau) - f(x_2, \tau)\| \leqslant K_1 \|x_1 - x_2\|, \qquad x, x_1, x_2 \in G, \qquad \tau \in R,$$

where G is a subset of a Banach space X which contains the range of f. He then observes for $\{f^i\}_{i=0}^{\infty} \subset f_1(G, K_1)$ that if G is open and solutions are unique, then:

$\|\phi(x, f_\tau^i, t) - \phi(x, f_\tau^0, t)\| \to 0$ uniformly for $t \in [0, T]$ as

$$\sup_{\left\{\substack{\tau_1 \in R, \\ \tau_1 < \tau_2 \leqslant \tau_1 + 1, \\ Z \in G,}\right\}} \left\| \int_{\tau_1}^{\tau_2} f^i(Z, \tau) - f^0(Z, \tau)\, d\tau \right\| \to 0.$$

This result is very similar to the continuity of solutions with respect to $\mathscr{I}_f$.

We require that for $\{f_i\}_{i=0}^{\infty} \subset \mathscr{I}_f$, and A a compact set in W:

$\|\phi(x, f_i, t) - \phi(x, f_0, t)\| \to 0$ uniformly on $[0, T]$ if for each compact set I in R,

$$\sup_{\substack{t \in I \\ x \in A}} \left\| \int_0^t f_i(x, \tau) - f_0(x, \tau)\, d\tau \right\| \to 0 \qquad \text{as} \quad i \to \infty.$$

Onuchic [16] considers continuous functions which have the property that for every compact set $A \subset W$ and each continuous function $x(t)$ contained in A on $[0, \infty)$

$$\int_S^{S+t} f(x(v), v)\, dv \to 0 \qquad \text{as} \quad S \to \infty \text{ uniformly for } t \in [0, 1]. \qquad (*)$$

It can be shown that if f satisfies the conditions given below for Theorem 1, then (*) is equivalent to the conclusion $f_\tau \to 0$ in $\mathscr{I}_f$ as $\tau \to \infty$. Onuchic used the property (*) to obtain stability results which are improved upon in this paper.

4. Development of the Semiflow

Theorem 1 gives sufficient conditions under which π generates a local semiflow on $\mathscr{I}_f \times W$. In order to obtain invariance we will need to know also that $\mathscr{I}_f$ is compact. Under conditions only slightly stronger than the conditions of Theorem 1 for continuity, it is shown in the next section (Theorem 2) that $\mathscr{I}_f$ is compact. These sufficient conditions are easy to verify and useful for applications.

Theorem 1. *Suppose for each compact set A in W that there exists a finite M_A and a locally L_1 function $K_A(t)$ such that for all x and y in A and t almost everywhere in R^+, $|f(x, t)| \leqslant M_A$, $|f(x, t) - f(y, t)| \leqslant K_A(t) \mid x - y \mid$ and $\int_\tau^{\tau+1} K_A(t)\, dt \leqslant M_A$ for all τ in R^+. Then π is continuous on $\mathscr{I}_f \times W \times R^+$ and property* (S4) *holds for π.*

It is easy to see that the condition on $K_A(t)$ is sufficient to exclude the example (20). However, it is not certain that the theorem would be false if the local bound on f were weakened in some way. It is also not certain that $\mathscr{I}_f$ is complete under these conditions. Completeness is needed to establish that $\mathscr{I}_f$ is compact but it is not required for the above continuity result.

Proof. Let $\{f_n\}$ in $\mathscr{I}_f$, $\{x_n\}$ in W, and $\{t_n\}$ in R^+ be such that in their respective topologies $f_n \to f_0$, $x_n \to x_0$ and $t_n \to t_0$ as $n \to \infty$. It is also assumed for each n that t_n is in the maximum interval $[0, \gamma_n)$ of $\phi(x_n, f_n, t)$. Let g_n denote the t_n translate of f_n with g_0 the t_0 translate of f_0. Since the domain and range of π are metric spaces, it is sufficient to prove, if t_0 is in the maximal domain of definition of the solution $\phi(x_0, f_0, t)$ to (5), that $g_n \to g_0$ in $\mathscr{J}_f$ as $n \to \infty$, lim inf $\alpha(f_n, x_n) \geqslant t_0$, and $|\phi(x_n, f_n, t_n) - \phi(x_0, f_0, t_0)| \to 0$ as $n \to \infty$.

To prove that $g_n \to g_0$ as $n \to \infty$, let h_n denote the t_n translate of f_0 and note that $\rho(g_n, g_0) \leqslant \rho(g_n, h_n) + \rho(h_n, g_0)$. Miller and Sell [4] prove that if f satisfies the weaker conditions (T1) and (T2) then the mapping from $R^+ \to G_1^*$ defined by f_t is a continuous function of t. Since $\mathscr{J}_f$ is a weaker topology than G_1^*, it follows that f_t is also continuous in t in the $\mathscr{J}_f$ topology. Therefore, since $t_n \to t_0$ as $n \to \infty$, it follows that $\rho(h_n, g_0) \to 0$ as $n \to \infty$. Also, $t_n \to t_0$ implies that $\{t_n\}$ is a bounded set with upper bound T. Now $f_n \to f_0$ in $\mathscr{J}_f$ implies for any $\tau > 0$ that

$$\sup_{0 \leqslant t \leqslant \tau + T} \int_0^t [f_n(x, s) - f_0(x, s)]\, ds \to 0 \tag{21}$$

uniformly for x in compact sets in W. It then follows that

$$\sup_{0 \leqslant t \leqslant \tau} \int_0^t [g_n(x, s) - h_n(x, s)]\, ds \to 0 \tag{22}$$

uniformly for x in compact subsets of W since

$$\begin{aligned} &\int_0^t [g_n(x, s) - h_n(x, s)]\, ds \\ &\quad = \int_0^t [f_n(x, s + t_n) - f_0(x, s + t_n)]\, ds = \int_0^{t+t_n} [f_n(x, s) - f_0(x, s)]\, ds \\ &\qquad - \int_0^{t_n} [f_n(x, s) - f_0(x, s)]\, ds, \end{aligned}$$

and each of the last two integrals $\to 0$ uniformly for x in compact sets as $n \to \infty$ since $t + t_n \leqslant \tau + T$. Therefore, $\rho(g_n, h_n) \to 0$ as $n \to \infty$ and hence $g_n \to g_0$ in $\mathscr{J}_f$ as $n \to \infty$.

For the proof that $\liminf_{n\to\infty} \alpha(f_n, x_n) \geqslant t_0$ and $\phi(x_n, f_n, t_n) \to \phi(x_0, f_0, t_0)$, let $[0, \gamma_0)$ be the maximal interval of definition for $\phi_0(t) \equiv x_0 + \int_0^t f_0(\phi_0(s), s)\, ds$ and let $\phi_n(t) \equiv \phi(x_n, f_n, t)$, $n = 0, 1,...$ on the maximal interval $[0, \gamma_n)$. The proof consists in showing that for $\beta < \gamma_0$, there exists an N such that $n \geqslant N$ implies that $\gamma_n > \beta$ and $\phi_n(t) \to \phi_0(t)$ uniformly on $[0, \beta]$. Since ϕ_0 is continuous, it will follow that $\phi_n(t_n) \to \phi_0(t_0)$.

Let $\hat{A}$ = the range of ϕ_0 on $[0, \beta]$. Since ϕ_0 is continuous, $\hat{A}$ is compact in W. Also since W is open, there exists a compact set A in W such that

$\hat{A} \subset$ interior A. Hence there exists an $\epsilon_0 > 0$ such that $|y - \phi_0(t)| \leqslant \epsilon_0$ for any t in $[0, \beta]$ implies that $y \in A$. Since A is compact in W, there is an M_A and a locally L_1 function $K_A(t)$ such that, if x and y are in A and t is in R^+, then $|f(x, t)| \leqslant M_A t$ a.e.; $|f(x, t) - f(y, t)| \leqslant K_A(t) \mid x - y \mid$; and

$$\int_\tau^{\tau+1} K_A(s)\, ds \leqslant M_A \qquad \text{for all} \quad \tau \quad \text{in} \quad R^+.$$

The next step is to show for x in A that $|f_n(x, t)| \leqslant M_A$ for t a.e. in R^+ for $n = 0, 1, \ldots$. Since $f_n \in \mathscr{J}_f$, f_n is an integrable function and equals the time derivative of its time integral for each x, a.e. in t. Therefore, given an arbitrary $\epsilon > 0$ and x in A, except on a set of measure zero, there exists a $d(\epsilon, x, t) > 0$ such that

$$\left| f_n(x, t) - \frac{1}{d(\epsilon, x, t)} \int_t^{t+d(\epsilon,x,t)} f_n(x, s)\, ds \right| < \frac{\epsilon}{2}. \tag{24}$$

Using this d, gives

$$|f_n(x, t)| \leqslant \left| f_n(x, t) - \frac{1}{d} \int_t^{t+d} f_n(x, s)\, ds \right| + \frac{1}{d} \left| \int_t^{t+d} (f_n(x, s) - f(x, s + \tau))\, ds \right| + \frac{1}{d} \left| \int_t^{t+d} f(x, s + \tau)\, ds \right|. \tag{25}$$

The first term is less than $\epsilon/2$. Since $f_n \in \mathscr{J}_f$, there exists some τ such that (for fixed $x \in A$, fixed $\epsilon > 0$) the second term on the right side of (18) is less than $\epsilon/2$. The third term is less than or equal to M_A from the bound on f. Since $\epsilon > 0$ is arbitrary, the bound on f_n is verified.

Let N_1 be such that for $n \geqslant N_1$, $|x_n - x_0| < (\epsilon_0/2)$ and choose $\sigma = \min\{\beta, (\epsilon_0/4M_A), 1\}$. It will be shown that on $[0, \sigma]$, if $n \geqslant N_1$, then $\phi_n(t) \in A$. This will follow by showing that $|\phi_n(t) - \phi_0(t)| < \epsilon_0$ for $t \in [0, \sigma]$.

$|\phi_n(0) - \phi_0(0)| = |x_n - x_0| < (\epsilon_0/2)$. Since each ϕ_n is continuous, $|\phi_n(t) - \phi_0(t)| < \epsilon_0$ on some interval (which may depend on n). Suppose there is a ξ such that $|\phi_n(t) - \phi_0(t)| < \epsilon_0$ for $0 \leqslant t < \xi \leqslant \sigma$ and $|\varphi_n(\xi) - \varphi_0(\xi)| = \epsilon_0$. Then

$$|\phi_n(\xi) - \phi_0(\xi)| \leqslant |x_n - x_0| + \left| \int_0^\xi f_n(\phi_n(s), s)\, ds \right| + \left| \int_0^\xi f_0(\phi_0(s), s)\, ds \right| < \frac{\epsilon_0}{2} + 2\sigma M_A \leqslant \epsilon_0 \quad \text{if } n \geqslant N_1.$$

Hence no such ξ exists when $n \geqslant N_1$ and $|\phi_n(t) - \phi_0(t)|$ must be less than ϵ_0 on $[0, \sigma]$.

For the next step suppose that $f_n = f_{\tau_n}$ for some $\tau_n \in R^+$ for each

$n = 1, 2, \ldots$. Then since $\phi_n(t) \in A$ for $0 \leqslant t \leqslant \sigma$ and $n \geqslant N_1$,

$$|\phi_n(t+h) - \phi_n(t)| = \left| \int_t^{t+h} f(\phi_n(s), s + \tau_n)\, ds) \right| \leqslant hM_A, \tag{27}$$

for all n and $0 \leqslant t \leqslant t + h \leqslant \sigma$. Therefore $\{\phi_n\}_{n=1}^{\infty}$ is a bounded equicontinuous family and by Ascoli's theorem, there exists a subsequence $\{\phi_{n_k}\}_{k=1}^{\infty}$ which converges uniformly to some function ϕ on $[0, \sigma]$. It will be shown that $\phi = \phi_0$ by using uniqueness of the solutions $\phi(x_0, f_0, \cdot)$ and showing that

$$\phi(t) = x_0 + \int_0^t f_0(\phi(s), s)\, ds \tag{28}$$

on $[0, \sigma]$. Using $\{\phi_n\}$ for the convergent subsequence gives:

$$\begin{aligned} \left| \phi(t) - x_0 - \int_0^t f_0(\phi(s), s)\, ds \right| &\leqslant |\phi(t) - \phi_n(t)| + |x_n - x_0| \\ &\quad + \left| \int_0^t [f_n(\phi_n(s), s) - f_0(\phi(s), s)]\, ds \right|. \end{aligned} \tag{29}$$

For an arbitrary $\epsilon > 0$, the uniform convergence of the subsequence (denoted by $\{\phi_n\}$) to ϕ implies that there exists an N_2 such that $n \geqslant N_2$ implies that each of the first two terms above are less than $\epsilon/4$ for all $t \in [0, \sigma]$.

Since $f_0 \in \mathscr{I}_f$ implies that f_0 satisfies (T1) and (T2), it follows that f_0 has a locally L_1 Lipschitz coefficient $K_0(s)$ on A. Let

$$\eta = \min \left\{ \epsilon/12M_A, \epsilon\Big/\left(6 \int_0^1 K_0(s)\, ds\right) \right\}.$$

Then there exists a piecewise constant function

$$y(s) = \sum_{i=1}^{J} \chi_i(s) \tag{30}$$

from $[0, \sigma]$ into A such that $\chi_i(s) = y_i$ on $[t_i, t_{i+1})$, $= 0$ elsewhere and $|y(s) - \phi(s)| < \eta$ on $[0, \sigma]$. Now choose N_3 such that for $n > N_3$, $|\phi_n(s) - \phi(s)| < \eta$ on $[0, \sigma]$ and choose N_4 such that for $n > N_4$, $| \int_0^t [f_n(x, s) - f_0(x, s)\, ds | < \epsilon/12J$ for all x in A and t in $[0, \sigma]$. The existence of N_4 follows from the convergence of f_n to f_0 in $\mathscr{I}_f$. Then

$$\begin{aligned} \left| \int_0^t [f_n(\phi_n(s), s) - f_0(\phi(s), s)]\, ds \right| &\leqslant \int_0^t |f_n(\phi_n(s), s) - f_n(y(s), s)|\, ds \\ &\quad + \left| \int_0^t [f_n(y(s), s) - f_0(y(s), s)]\, ds \right| \\ &\quad + \int_0^t |f_0(y(s), s) - f_0(\phi(s), s)|\, ds. \end{aligned} \tag{31}$$

For $0 \leqslant t \leqslant \sigma$, since $\sigma \leqslant 1$ and $f_n = f_{\tau_n}$, the Lipschitz condition on f implies that the first term is $\leqslant M_A \sup_{0 \leqslant s \leqslant \sigma} | \phi_n(s) - y(s)| \leqslant M_A \sup_{0 \leqslant s \leqslant \sigma} \{| \phi_n(s) - \phi(s)| + | \phi(s) - y(s)|\} < 2\eta M_A \leqslant \epsilon/6$ for all n sufficiently large. Similarly $| f_0(y(s), s) - f_0(\phi(s), s)| \leqslant K_0(s)\eta$ from which it follows that the third term on the right side of (31) is $< \epsilon/6$. The second term is $\leqslant$

$$\sum_{i=1}^{J} \int_{t_i}^{t_{i+1}} [f_n(y_i, s) - f_0(y_i, s)]\, ds$$

$$\leqslant 2 \sum_{i=1}^{J} \sup_{t \in [0, \sigma]} \left| \int_0^t [f_n(y_i, s) - f_0(y_i, s)]\, ds \right| < \frac{\epsilon}{6} \tag{32}$$

for all n sufficiently large. Putting everything together in (29) gives $| \phi(t) - x_0 - \int_0^t f_0(\phi(s), s)\, ds | < \epsilon$ for all t in $[0, \sigma]$. Since ϵ is arbitrary it follows from the uniqueness argument that $\phi = \phi_0$.

Similarly it follows that every subsequence of ϕ_n must, in turn, have a subsequence which converges uniformly to ϕ_0. Therefore, the original sequence $\{\phi_n\}$ converges uniformly to ϕ_0 on $[0, \sigma]$.

If $\sigma < \beta$, then the above procedure can be repeated starting from $t_0 = \sigma$ and establishing that $\phi_n \to \phi_0$ uniformly on $[\sigma, \sigma_2]$ with $\sigma_2 = \min\{\beta, 2\sigma\}$. Continuing it is apparent that β will be reached in a finite number of steps and it will follow that $\sigma_n \to \sigma_0$ as $n \to \infty$ uniformly on $[0, \beta]$ and since β is arbitrary and less than γ_0, the theorem has been proved under the supposition that each f_n is in $\{f_t\}$, $0 < t < \infty$.

In the general case it follows from the definition of $\mathscr{J}_f$ that for each $f_n \in \mathscr{J}_f$ there exists a sequence $\{\tau(n, k)\}_{k=1}^{\infty}$ such that $f_{\tau(n,k)} \to f_n$ in $\mathscr{J}_f$ as $k \to \infty$. Let $A^1 = \{x \in A \text{ such that } \operatorname{dist}\{x, \hat{A}\} \leqslant \epsilon_0/2\}$. Clearly this is a compact subset of W with $\hat{A}$ in its interior. In the same way that has been used to establish that ϕ_n is in A on $[0, \sigma]$ for $n > N_1$, it follows that for $n > N_5$, $\phi_n(t)$ is contained in A^1 for t in $[0, \sigma/2]$. Therefore, proceeding along the lines of the above proof for each fixed n, it follows that if $| y - \phi_n(t)| \leqslant \epsilon_0/2$ for some $t \in [0, \sigma/2]$ and for some $n > N_5$, then $y \in A$.

Defining appropriate piecewise constant functions y_n in A and using $\epsilon_0/2$ in place of ϵ_0 and $\sigma/2$ in place of σ gives $\phi_{(n,k)} \to \phi_n$ as $k \to \infty$ uniformly on $[0, \sigma/2]$ for each $n > N_5$, where $\phi_{(n,k)}(t)$ is the solution to $\phi_{(n,k)}(t) = x_n + \int_0^t f_{\tau(n,k)}(\phi_{(n,k)}(s), s)\, ds$.

Since $f_{\tau(n,k)} \to f_n$ as $k \to \infty$ and $f_n \to f_0$ as $n \to \infty$ a diagonalized sequence $\{f_{\tau(n,k_n)}\}_{n=1}^{\infty}$ can be found such that $| \phi_{(n,k_n)} - \phi_0 | \to 0$ uniformly on $[0, \sigma/2]$ as $n \to \infty$. This can be done by choosing each k_n large enough so that $\rho(f_{\tau(n,k_n)}, f_n) < 1/n$ and $| \phi_{(n,k_n)}(t) - \phi_n(t)| < 1/n$ for all t in $[0, \sigma/2]$. Then since each $f_{\tau(n,k_n)}$ is in f_t, the previous proof shows that $\phi_{(n,k_n)} \to \phi_0$ uniformly on $[0, \sigma/2]$. Since $| \phi_n - \phi_0 | \leqslant | \phi_n - \phi_{(n,k_n)} | + | \phi_{(n,k_n)} - \phi_0 |$, it follows that $\phi_n \to \phi_0$ uniformly on $[0, \sigma/2]$. Repeating the above procedure in $\sigma/2$ sized steps gives the result that $\phi_n \to \phi_0$ uniformly on $[0, \beta]$ and the proof is complete.

COROLLARY 1. *If f satisfies the conditions of Theorem 1, then π is a local semiflow on $W \times \mathscr{J}_f \times R^+$.*

The proof follows from the definition of a local semiflow, Theorem 1, and the fact [4] that solutions are uniquely defined on maximal intervals.

5. ESTABLISHING COMPACTNESS OF $\mathscr{J}_f$

In order to obtain the desired invariance result for Eq. (5), we need to know that $\mathscr{J}_f$ is compact. Useful sufficient conditions for this compactness are given in Theorem 2. Compactness will be established by showing that every infinite sequence in $\mathscr{J}_f$ has a Cauchy subsequence and that every Cauchy sequence in $\mathscr{J}_f$ has a limit point in $\mathscr{J}_f$ (that is, $\mathscr{J}_f$ is complete.). It will be sufficient to show for any sequence $\{t_n\}_{n=1}^{\infty}$ in R^+ that $\{f_{t_n}\}_{n=1}^{\infty}$ has a Cauchy subsequence and that if $\{f_{t_n}\}_{n=1}^{\infty}$ is Cauchy in $\mathscr{F}$, then there exists a function g in $\mathscr{F}$ such that $f_{t_n} \to g$ in $\mathscr{F}$ as $n \to \infty$. Thus every sequence has a Cauchy subsequence and every Cauchy sequence converges. Therefore, $\{f_t: t \in R^+\}$ is relatively compact in $\mathscr{F}$ and it follows that its closure $\mathscr{J}_f$ is compact.

THEOREM 2. *Assume that for each x, $f(x, t)$ is a measurable function of t and that for each compact set A in W, there exists an M_A such that properties* (C1) *and* (C2) *below are satisfied.*

Then $\mathscr{J}_f$ is compact and every function in $\mathscr{J}_f$ is equivalent in $\mathscr{F}$ to a function which satisfies (C1) *and* (C2) *with the same bounds.*

(C1) $|f(x, t)| \leqslant M_A$ *for all x in A and t a.e. in R^+*

(C2) $|f(x, t) - f(y, t)| \leqslant M_A |x - y|$ *for all x and y in A and t a.e. in R^+.*

Proof. (i) Proof for an arbitrary sequence $\{t_n\}$ in R^+, that $\{f_{t_n}\}$ has a subsequence which is Cauchy in $\mathscr{F}$.

Define

$$F_n(x, t) = \int_0^t f_{t_n}(x, s)\, ds = \int_0^t f(x, s + t_n)\, ds. \tag{33}$$

For any compact set A in W, the bound M_A on f implies that $\{F_n\}_{n=1}^{\infty}$ is an equicontinuous family of functions from $A \times R^+$ into R^n and for each (x, t), $\{F_n(x, t): 0 \leqslant n < \infty\}$ is bounded. By Ascoli's Theorem there exists a subsequence $\{F_{n_k}\}_{k=1}^{\infty}$ which converges pointwise on $A \times R^+$ and the convergence is uniform on compact subsets of R^+.

Since W is in R^n, it can be covered by a countable collection of compact sets. Then a diagonalization procedure will give a subsequence of $\{F_n\}$ which converges uniformly on all compact subsets of $W \times R^+$. The related subsequence of $\{f_{t_n}\}$ is then easily seen to be a Cauchy sequence in $\mathscr{I}$.

Let $\{\alpha_n\}$ be a countable dense subset of W. For each α_n, let $r_n = 1$ if $W = R^n$ and if not let $r_n = \{\inf | \alpha_n - x|$: for x outside $W\}$. Let $A_n =$ the closed sphere of radius $r_n/2$ centered at α_n. Clearly A_n is in W and A_n is compact for all n. Now suppose p is a point in W which is not in any of the sets A_n. Then since $\{\alpha_n\}$ is dense in W, there exists a subsequence $\{\alpha_{n_k}\}$ which converges to p as $k \to \infty$. Since p is not in A_{n_k}, $| \alpha_{n_k} - p | > r_{n_k}/2$ and since $\alpha_{n_k} \to p$ as $k \to \infty$, r_{n_k} must $\to 0$ as $k \to \infty$. This cannot be true if $W = R^n$. If $W \neq R^n$, for each $k = 1, 2, \ldots$, there must exist a point x_{n_k} outside W such that $| \alpha_{n_k} - x_{n_k} | \leqslant r_{n_k} + (1/k)$. Therefore, $| p - x_{n_k} | \leqslant | p - \alpha_{n_k} | + | \alpha_{n_k} - x_{n_k} | \to 0$ as $k \to \infty$. Since W is open, the complement is closed and $x_{n_k} \to p$ as $k \to \infty$ implies that p is not in W, a contradiction. It follows that

$$\bigcup_{n=1}^{\infty} A_n = W.$$

Let $\{F_{1,n}(x, t)\}_{n=1}^{\infty}$ be a subsequence of $\{F_n\}$ that converges pointwise on $A_1 \times R^+$ and uniformly on compact subsets. By induction for each integer m, let $\{F_{m,n}(x, t)\}_{n=1}^{\infty}$ be a subsequence of $\{F_{m-1,n}(x, t)\}$ which converges uniformly on compact subsets of $A_m \times R^+$. Now consider the sequence $\{F_{i,i}(x, t)\}_{n=1}^{\infty}$. Since for $i \geqslant m$ this sequence is a subsequence of $\{F_{m,n}(x, t)\}_{n=1}^{\infty}$, the sequence converges for each A_m and uniformly on compact sets in $A_m \times R^+$. For any compact set B in W, B is covered by the union of the open interiors of the A_n's (which also cover W), therefore, B has a finite subcover or

$$B \subset \bigcup_{k=1}^{m} A_{n_k}$$

and it follows that the sequence $\{f_{t_{i,i}}\}_{i=1}^{\infty}$ for which the $F_{i,i}$ functions are the integrals, is a Cauchy subsequence of $\{f_{t_n}\}_{n=1}^{\infty}$ in $\mathscr{I}$.

(ii) Proof of Completeness. Suppose that $\{t_n\}_{n=1}^{\infty}$ is a sequence in R^+ such that for (x, t) in any compact set B in $W \times R^+$,

$$\int_0^t [f_{t_n}(x, s) - f_{t_m}(x, s)]\, ds \to 0$$

uniformly on B as $m, n \to \infty$. The rest of the proof consists in establishing that there exists a function g in $\mathscr{I}$ such that $\int_0^t [f_{t_n}(x, s) - g(x, s)]\, ds \to 0$ uniformly for (x, t) in any compact set in $W \times R^+$ as $n \to \infty$, and such that g satisfies (C1) and (C2) with the same bounds M_A.

If $\{f_{t_n}\}$ is Cauchy in $\mathscr{I}$, then the functions $F_n(x, t)$ are Cauchy in the complete metric space $\mathscr{F}$. Therefore, $F_n(x, t)$ approaches some continuous function $G(x, t)$ uniformly on compact subsets of $W \times R^+$ as $n \to \infty$.

The rest of the proof consists in showing first that G is absolutely continuous in t for each x and therefore at each point x, G has a derivative g with respect to t at almost all t. The function g can then be defined at points where G is not differentiable with respect to t in such a way that g is continuous in x for each t and satisfies the same bounds and Lipschitz bounds that f does.

Let I be some finite interval in R^+ and let x be some fixed point in W. Then for any finite collection $(\tau_1, \tau_1', \tau_2, \tau_2', \ldots, \tau_N, \tau_N')$ of points in I.

$$\sum_{i=1}^{N} | G(x, \tau_i') - G(x, \tau_i)|$$

$$\leqslant \sum_{i=1}^{N} \{| G(x, \tau_i) - F_n(x, \tau_i)|$$

$$+ | G(x, \tau_i') - F_n(x, \tau_i')| + |F_n(x, \tau_i') - F_n(x, \tau_i)|\}. \quad (34)$$

For any fixed $\epsilon > 0$, choose n_ϵ such that $| F_{n_\epsilon}(x, t) - G(x, t)| < \epsilon/3N$ for all t in I. Then

$$|F_n(x, \tau_i') - F_n(x, \tau_i)| = \left| \int_{t_i}^{t_i'} f(x, s + t_n)\, ds \right| \leqslant M_x | \tau_i' - \tau_i | \quad (35)$$

where M_x is a t a.e. bound for $f(x, \cdot)$. Putting these results together gives

$$\sum_{i=1}^{N} | G(x, \tau_i') - G(x, \tau_i)| < \frac{2\epsilon}{3} + M_x \sum_{i=1}^{N} | \tau_i' - \tau_i | \quad (36)$$

and if the sum is less than $\epsilon/(3M_x)$, then

$$\sum_{i=1}^{N} | G(x, \tau_i') - G(x, \tau_i)| < \epsilon.$$

Therefore, G is absolutely continuous in t for each x and there exists a function g such that

$$G(x, t) = \int_0^t g(x, s)\, ds$$

where g is the time derivative of G at almost all t.

Let $\{\alpha_n\}_{n=1}^{\infty}$ be a countable dense set in W. For each point α_k in $\{\alpha_n\}$ and each function f_{t_n}, let $A_{(k,n)} = \{t \in R^+ : F_n(\alpha_k, t)$ is not differentiable with respect to t}. Let $B_k = \{t \in R^+: G(\alpha_k, t)$ is not t differentiable}. Then let

$$A_0 = \left(\bigcup_{n,k=1}^{\infty} A_{(k,n)}\right) \cup \left(\bigcup_{k=1}^{\infty} B_k\right).$$

The set A_0 has Lebesgue measure zero since A_0 is the union of a countable collection of sets which each have measure zero. For $t \in A_0$ define $g(x, t) \equiv 0$ for all x in W and for t not in A_0, define $g(\alpha_k, t) = \partial G(\alpha_k, t)/\partial t$. The next step is to let $g(x, t) = \lim g(\alpha_k, t)$ as $\alpha_k \to x$ for an appropriate subsequence of the original sequence $\{\alpha_n\}$. However, in order for this definition to be unique for any subsequence which converges to x, it must be shown that g is continuous on $\{\alpha_n\}$ for each t. This result is obvious for $t \in A_0$. Note that g is continuous on $\{\alpha_n\}$ at some t if for any point α_k and any $\epsilon > 0$, there exists a neighborhood N_ϵ of α_k such that if α_i is in N_ϵ, then $|g(\alpha_k, t) - g(\alpha_i, t)| < \epsilon$. Actually since Lipschitz continuity will be required, the following proof establishes that for any compact set N in W, there exists a number M_N such that if α_j and α_k are in N, then $|g(\alpha_j, t) - g(\alpha_k, t)| \leqslant M_N |\alpha_j - \alpha_k|$ and furthermore, M_N is independent of t.

Let N be a compact set in W (which may have a nonempty interior which can be used as a neighborhood for one of the α_n's). Let α_j and α_k be two arbitrary points from $\{\alpha_n\}$ in N. Then

$$\begin{aligned}
|g(\alpha_j, t) - g(\alpha_k, t)| \leqslant \\
&\left| g(\alpha_j, t) - \frac{G(\alpha_j, t+d) - G(\alpha_j, t)}{d} \right| \\
&+ \left| g(\alpha_k, t) - \frac{G(\alpha_k, t+d) - G(\alpha_k, t)}{d} \right| + \frac{1}{d}\{| G(\alpha_j, t) - F_n(\alpha_j, t)| \\
&+ | G(\alpha_k, t) - F_n(\alpha_k, t)| + | G(\alpha_j, t+d) - F_n(\alpha_j, t+d)| \\
&+ | G(\alpha_k, t+d) - F_n(\alpha_k, t+d)|\} \\
&+ \frac{1}{d} | F_n(\alpha_j, t+d) - F_n(\alpha_j, t) - F_n(\alpha_k, t+d) + F_n(\alpha_k, t)|.
\end{aligned}$$

Since G is t differentiable at α_j and α_k (for t outside A_0), given any $\epsilon > 0$, there exists a $d > 0$ such that each of the first two terms above are less than $\epsilon/3$. With t and d fixed, the convergence of $\{F_n\}$ to G implies that an n can be found such that the expression in brackets is less than $\epsilon d/3$. The final term can be rewritten as

$$\frac{1}{d} \left| \int_t^{t+d} [f(\alpha_j, s + t_n) - f(\alpha_k, s + t_n)] \, ds \right|.$$

But for s a.e.

$$|f(\alpha_j, s) - f(\alpha_k, s)| \leqslant M_N |\alpha_j - \alpha_k|,$$

where M_N is the bound for f on N as specified in the conditions of Theorem 2.

Putting these results together gives

$$|g(\alpha_k, t) - g(\alpha_j, t)| \leqslant \epsilon + M_N |\alpha_k - \alpha_j| \tag{38}$$

for all t in R^+. Therefore, since $\epsilon > 0$ is arbitrary it can be removed and since $\{\alpha_n\}$ is dense in W, g is now uniquely defined on $W \times R^+$.

If x is in the interior of a compact set N in W, then $|g(x, t)| \leqslant \sup |g(\alpha_k, t)|$ for $\alpha_k \in N$. A nonempty interior is needed to assure that there are points in $\{\alpha_n\}$ arbitrarily close to x. Note that

$$\begin{aligned} |g(\alpha_k, t)| \leqslant & \left| g(\alpha_k, t) - \frac{G(\alpha_k, t+d) - G(\alpha_k, t)}{d} \right| \\ & + \frac{1}{d}\{|G(\alpha_k, t+d) - F_n(\alpha_k, t+d)| + |G(\alpha_k, t) - F_n(\alpha_k, t)|\} \\ & + \frac{1}{d}\int_t^{t+d} |f(\alpha_k, s+t_n)|\, ds. \end{aligned} \tag{39}$$

For arbitrary $\epsilon > 0$, a d can be found such that the first term is less than $\epsilon/2$ and as before, an n can be found such that the expression in brackets is less than $\epsilon d/2$. The final term is less than M_N the bound for f on N. Since ϵ is arbitrary, the result $|g(x, t)| \leqslant M_N$ is obtained. Since any compact set in W can be contained in the interior of another compact set in W, it follows that g is uniformly bounded on compact sets in W.

Similarly if x and y are in a compact set in W, they are in the interior of a compact set N contained in W and

$$\begin{aligned} |g(x, t) - g(y, t)| \leqslant & |g(x, t) - g(\alpha_k, t)| + |g(y, t) - g(\alpha_m, t)| \\ & + |g(\alpha_k, t) - g(\alpha_m, t)|. \end{aligned} \tag{40}$$

For arbitrary $\epsilon > 0$, choose α_k and α_m from $\{\alpha_n\}$ such that α_k and α_m are in N and such that $|\alpha_k - x| < \epsilon/6M_N$, $|\alpha_m - y| < \epsilon/6M_N$, $|g(x, t) - g(\alpha_k, t)| < \epsilon/3$ and $|g(y, t) - g(\alpha_m, t)| < \epsilon/3$. Such points α_k and α_m exist since $\{\alpha_n\}$ is dense in W and $|g(\alpha_i, t) - g(x, t)| \to 0$ as $\alpha_i \to x$ and since x and y are in open subsets of N, for α_i close enough to x, α_j close enough to y, α_i and α_j are in N. Using $|g(\alpha_k, t) - g(\alpha_m, t)| \leqslant M_N |\alpha_k - \alpha_M|$ gives:

$$\begin{aligned} |g(x, t) - g(y, t)| & < 2\epsilon/3 + M_N |\alpha_k - \alpha_m| \\ & \leqslant 2\epsilon/3 + M_N\{|\alpha_k - x| + |\alpha_m - y| + |x - y|\} \\ & < \epsilon + M_N |x - y| \end{aligned} \tag{41}$$

and since ϵ was arbitrary, $|g(x, t) - g(y, t)| \leqslant M_N |x - y|$. For a compact set A with no interior and a compact set N such that A is contained in the interior of N,

$$|g(x, t)| \leqslant |g(x, t) - g(\alpha_k, t)| + |g(\alpha_k, t)|. \tag{42}$$

For an arbitrary $\epsilon > 0$, an α_k in N can be found such that $|\alpha_k - x| < \epsilon/3M_N$ and $|g(x, t) - g(\alpha_k, t)| < \epsilon/3$, and an n and d can be found such that

$$\begin{aligned} |g(\alpha_k, t)| &\leqslant \frac{\epsilon}{3} + \frac{1}{d}\int_t^{t+d} |f(\alpha_k, s + t_n)|\, ds \\ &\leqslant \frac{\epsilon}{3} + \frac{1}{d}\left\{\int_t^{t+d} M_N |\alpha_k - x|\, ds\right\} + \int_t^{t+d} \{f(x, s + t_n)|\, ds\} \\ &\leqslant \frac{\epsilon}{3} + M_N |\alpha_k - x| + M_A \end{aligned} \tag{43}$$

where M_A is the bound for f on A. Therefore, $|g(x, t)| \leqslant M_A + \epsilon$ and since ϵ is arbitrary it can be removed. Similarly for x, y in A there are points α_j, α_k in N such that $\{|\alpha_j - x| + |\alpha_k - y|\} < \epsilon/4M_N$,

$$\begin{aligned} |g(x, t) - g(y, t)| \leqslant |g(x, t) - g(\alpha_j, t)| &+ |g(y, t) - g(\alpha_k, t)| \\ &+ |g(\alpha_j, t) - g(\alpha_k, t)|, \end{aligned} \tag{44}$$

and the first two terms are each less than $\epsilon/4$. A d and n can be found such that

$$\begin{aligned} |g(\alpha_j, t) - g(\alpha_k, t)| &\leqslant \frac{\epsilon}{4} + \frac{1}{d}\int_t^{t+d} |f(\alpha_j, s + t_n) - f(\alpha_k, s + t_n)|\, ds \\ &\leqslant \frac{\epsilon}{4} + \frac{1}{d}\int_t^{t+d} M_N\{|\alpha_j - x| + |\alpha_k - y|\}\, ds \\ &\quad + \frac{1}{d}\int_t^{t+d} |f(x, s + t_n) - f(y, s + t_n)|\, ds \\ &< \frac{\epsilon}{2} + M_A |x - y|. \end{aligned} \tag{45}$$

Combining (44) and (45) and noting that $\epsilon > 0$ is arbitrary gives $|g(x, t) - g(y, t)| \leqslant M \ |x - y|$. Thus g is in $\mathscr{I}$ and g satisfies (C1) and (C2) with the same bounds satisfied by f and the proof is complete.

It is interesting to compare Theorem 2 with corresponding conditions for a compact motion in $\mathscr{F}$ or G_p^*. Sell [2] states in Theorem 14 that the set $\{f_t : t \in R^+\}$ is compact in $\mathscr{F}$ if and only if f is bounded and uniformly continuous on each set $M \times R^+$ where M is a compact subset of the domain W. Theorem 2 above is an extension in that no continuity conditions with respect to t are required, only a uniform Lipschitz continuity with respect to x on $M \times R^+$.

Sufficient conditions for compact motion in $G_p{}^*$ are given in [4, Theorem II.5]. These conditions which are difficult to verify are given below. Suppose $g \in G_p{}^*$, $1 \leqslant p < \infty$ and assume that for every compact set K in W, there exists a bound $m(t)$ and a Lipschitz coefficient $k(t)$ for g on K such that for some $B > 0$,

(i) $\int_t^{t+1} [m(s)]^p \, ds \leqslant B$ and $\int_t^{t+1} [k(s)]^p \, ds \leqslant B$ for all $t \geqslant 0$,

(ii) for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\int_t^{t+1} | m(s+h) - m(s)|^p \, ds \leqslant \epsilon$ and $\int_t^{t+1} | k(s+h) - k(s)|^p \, ds \leqslant \epsilon$ for all $t \geqslant 0$ whenever $| h | \leqslant \delta$, and

(iii) for every piece-wise continuous function x mapping R^+ into W and every $\epsilon > 0$, there is a $\delta > 0$ such that $\int_t^{t+1} | g(x(s+h), s+h) - g(x(s), s)|^p \, ds \leqslant \epsilon$ for all $t \geqslant 0$ whenever $| h | \leqslant \delta$.

Then the motion g_t is compact in G_r for $1 \leqslant r \leqslant p$.

Although condition (i) is slightly weaker than condition (C1), the example $g(x, t) = \sin e^t$ does not satisfy (iii).

Corollary 2. *If f satisfies the conditions of Theorem 2, then the positive motion of $\pi(f, x, t)$ is compact in $\mathscr{I} \times W$ if and only if the solution $\phi(x, f, t)$ exists on $[0, \infty)$ and is contained in a compact subset of W.*

6. Applications

Many of the theorems obtained by Sell [3] follow from these results with the same proofs given by Sell except for the change in the topology used, and Peng's results can be obtained and generalized. It is assumed below that f satisfies the conditions of Theorem 1 so that π generates a semiflow.

Corollary 3. *If $\phi(x, f, t)$ remains in W for all t in R^+ and $\Omega(f, x)$ is the positive limit set of $\pi(f, x, t)$ in $\mathscr{I} \times W$, then if $\mathscr{I}_f$ is compact, a point $\hat{x}$ lies in the projection $L^+(f, x)$ of $\Omega(f, x)$ into W if and only if there exists a sequence $\{\tau_n\}_{n=1}^{\infty}$ such that $\tau_n \to \infty$ as $n \to \infty$ and $\phi(x, f, \tau_n) \to \hat{x}$ as $n \to \infty$.*

Corollary 4. *If $\phi(x, f, t)$ is contained in a compact set in W for $t \in [0, \infty)$ and $\mathscr{I}_f$ is compact, then $\Omega(f, x)$ in $\mathscr{I} \times W$ is nonempty, compact, and invariant under π. Therefore, if (f^*, x^*) is in $\Omega(f, x)$, then $\phi(x^*, f^*, t)$ is contained in the compact set $L^+(f, x)$ for $t \in (-\infty, \infty)$. Also, if $x^* \in L^+(f, x)$ then $\phi(x^*, f^*, t)$ is in $L^+(f, x)$ for all t in R for some function f^* in $\mathscr{I}_f$.*

Proof. First note that f^* can be defined on $(-\infty, \infty)$ by observing that $\rho(f_{\tau_n}, f^*) \to 0$ as $\tau_n \to \infty$ implies that for any t in R, there exists an N such that $n \geqslant N$ implies $\tau_n + t \in R^+$ so $f_{\tau_n}(x, t)$ is defined for all n sufficiently large. The properties of $\Omega(f, x)$ then follow from Hale [10], Lemma 3. Then for (f^*, x^*) in $\Omega(f, x)$, $\pi(f^*, x^*, t) = (f_t^*, \phi(x^*, f^*, t)) \in \Omega(f, x)$ for all t in R and $\phi(x^*, f^*, t) \in L^+(f, x)$. If $x^* \in L^+(f, x)$ then there exists a sequence $\tau_n \to \infty$ such that $\phi(x, f, \tau_n) \to x^*$ as $n \to \infty$. Using the compactness of $\mathscr{J}_f$ it follows that some subsequence $\pi(f, x, \sigma_n) = (f_{\sigma_n}, \phi(x, f, \sigma_n)) \to (f^*, x^*)$ as $n \to \infty$. Also for any $t \in R$, for $\sigma_n + t > 0$, $\pi(f, x, \sigma_n + t) = \pi(f_{\sigma_n}, \phi(x, f, \sigma_n), t) \to \pi(f^*, x^*, t)$ from the continuity of π. It follows from Corollary 3 that $\phi(x^*, f^*, t) \in L^+(f, x)$.

COROLLARY 5. *If $\phi(x, f, t)$ is in a compact set in W for $t \geqslant 0$ and $\mathscr{J}_f$ is compact, then for every point (f^*, x^*) in $\Omega(f, x)$, there exists a sequence $\{\tau_n\}_{n=1}^{\infty}$ in R^+ with $\tau_n \to \infty$ as $n \to \infty$ such that $\phi(x, f, \tau_n + t) \to \phi(x^*, f^*, t)$ uniformly on compact sets in R as $n \to \infty$.*

The above three corollaries and their proofs correspond to Lemma 2, Theorem 1, and Theorem 2 respectively in Sell's paper [3].

Peng's invariance property can be generalized as follows:

COROLLARY 6. *If $\phi(x, f, t)$ is in a compact set in W for $t \geqslant 0$ and $\mathscr{J}_f$ is compact, then the positive limit set $L^+(f, x)$ is quasi-invariant in a sense similar to that used by Peng [1] (that is, if $x^* \in L^+(f, x)$, then there exists an f^* and $\{\tau_n\}_{n=1}^{\infty}$ such that $f_{\tau_n} \to f^*$ in $\mathscr{J}$ as $n \to \infty$ and $\phi(x^*, f^*, t)$ is in $L^+(f, x)$ for all t in R and $\phi(x^*, f_{\tau_n}, t) \to \phi(x^*, f^*, t)$ uniformly on bounded sets in R as $n \to \infty$). In particular, if $\phi(x, f, \tau_n) \to x^*$ as $n \to \infty$ for some sequence $\{\tau_n\}$ which goes to infinity, then $f_{\sigma_k} \to f^*$ (in $\mathscr{J}$) as $k \to \infty$ for some subsequence $\{\sigma_k\}$ of the sequence $\{\tau_n\}$.*

Proof. The first part follows from Corollary 4. The characterization of f^* follows from the fact that since $\mathscr{J}_f$ is compact, $\{f_{\tau_n}\}$ contains a Cauchy subsequence which converges to some function f^* and for the subsequence $\{\sigma_k\}$, $\pi(f, x, \sigma_k) \to (f^*, x^*)$ as $k \to \infty$.

This result combined with Theorem 2 includes Peng's result since the class of functions he considers satisfy the conditions of Theorem 2. This can be shown since $f[x, q(t)]$ is continuous in (x, q) and C^1 in x. Also q is a measurable function from R (or R^+) into a compact set Q in R^n. Therefore, for a compact set A in W, $A \times Q$ is compact in the product topology and since f is continuous in (x, q), the range of f on $A \times Q$ is compact. Similarly $\partial f/\partial x$ is continuous in (x, q) on $A \times Q$ and, therefore, is bounded. It follows that $| f[x, q(t)] - f[y, q(t)] | \leqslant M_A | x - y |$ for all x, y in A and all t in R^+ where $M_A = \sup | \partial f/\partial x |$ on $A \times Q$.

This result generalizes Peng's result since many functions satisfy the conditions of Theorem 2 but do not have the form used by Peng. One example is: $f(x, t) = \epsilon^{-tx^2}$. Also Theorem 2 uses the weaker condition of Lipschitz continuity in x instead of C^1 continuity as required by Peng. Furthermore, the present result does not require Peng's condition (A1) that for each $T > 0$, and each bounded set B in R^n, $\phi(x, f, t) \in A(B, T)$ for all x in B and t in $[0, T]$ where $A(B, T)$ is some bounded set. The function $f(x, t) = x^2$ for example, does not satisfy this condition but it does satisfy the conditions of Theorem 2 and $\phi(x, f, t)$ is bounded on $[0, \infty)$ for $x \leqslant 0$.

Peng's result that if $f(x, q)$ is convex in R^n for each x, then $f^*(x, t) = f[x, g^*(t)]$ for some measurable function g^* with range in Q also follows with the weaker conditions on f.

7. A Generalized Invariance Principle for Nonautonomous Differential Equations and Further Applications

In this section we use the invariance property of Corollary 6 to obtain Liapunov-type results for studying the asymptotic behavior of solutions of nonautonomous systems.

Consider the set of differential equations

$$\dot{x} = f^\alpha(x, t), \qquad x(0) = x, \tag{46}$$

where α is a generalized index which ranges over some specified set Σ. A set N in R^n is called quasi-invariant in the positive direction (QIP) with respect to Σ if for each x in N there exists an α, a continuous function $\psi^\alpha(t)$ on R and a sequence $\{t_n\}_{n=1}^{\infty}$ with $t_n \to \infty$ as $n \to \infty$ such that $\phi(x, f_{t_n}^\alpha, t) \to \psi^\alpha(t)$ uniformly on bounded sets in R as $n \to \infty$ and $\psi^\alpha(t)$ is in N for all $t \in R$. In Peng's application the index is q and the set Σ is the set of admissible q's so that $f^q(x, t) = f(x, q(t))$. The above definition is slightly more restrictive than Peng's definition because Peng requires only that $\phi(x, f^{q_k}, t) \to \psi(t)$ as $k \to \infty$ for an arbitrary sequence of admissible functions $\{q_k\}$ rather than just for translates (which are also admissible) of a particular q. Furthermore, the limit function is in N for all t rather than just for $t \geqslant 0$ as required by Peng. Therefore, it is possible for a set to be quasi-invariant in the sense used by Peng but not in the more restrictive sense used here. It is usually preferable to use the most restrictive applicable invariance definition since this leads to stronger conditions on the limit set.

Theorem 3 below is the generalized invariance result for (46) and, when Σ contains only one element, this is a new invariance principle for non-autonomous ordinary differential equations.

THEOREM 3. *For each* α *in* Σ *assume that* (i) f *satisfies the conditions of Theorem* 2 (*not necessarily with the same bound* M_A *for a compact set* A *in* W) *and* (ii) *for each* x *in* H, *a subset of* W, *each solution* $\phi(x, f^\alpha, t)$ *of* (46) *that remains in* W *for* $t \geqslant 0$ *approaches a set* E *as* $t \to \infty$. *Then each solution that stays in a compact subset of* W *for* $t \geqslant 0$ *approaches* M *as* $t \to \infty$, *where* M *is the largest set which is QIP relative to* Σ *in* E.

Proof. This follows from Corollary 6 and the observation that the union of quasi-invariant sets is quasi-invariant.

A set N in R^n is called invariant in the positive direction with respect to Σ if the function $\psi^\alpha(t) = \phi(x, f^\beta, t)$ for some β in Σ. A sufficient condition for the positive limit set to be invariant in the positive direction is that $\{f^\alpha\}$, $\alpha \in \Sigma$, include all functions in the positive limit set of f_t^α for each α and, in addition, each f^α satisfies the conditions of Theorem 2. If this is true then, since $\psi^\alpha(t) = \phi(x, f^*, t)$, where $f^* = \lim_{n\to\infty} f_{t_n}^\alpha$ and $t_n \to \infty$ it follows that $f^* = f^\beta$ for some β in Σ and therefore the positive limit set of $\phi(x, f^\alpha, t)$ will be invariant in the positive direction with respect to Σ. It can be shown that Peng's functions have this property if his set Q is convex. Other useful sufficient conditions have not been found.

Consider now the single nonautonomous differential equation satisfying the conditions of Theorem 2 on an open set W in R^n

$$\dot{x} = f(x, t). \tag{47}$$

Let G be an arbitrary set such that $\bar{G} \subset W$. Let V be a C^1 function from $G \times R^+$ to R^n such that for each point x in $\bar{G}$ there is a neighborhood N of x with V bounded from below in $(G \cap N) \times R^+$. Suppose that at each x in $\bar{G}$, for t a.e., $\partial V(x, t)/\partial x \cdot f(x, t) + \partial V(x, t)/\partial x \leqslant -W(x) \leqslant 0$ where W is continuous on $\bar{G}$. Then V is said to be a Liapunov function for (47) on G. It is easily shown that for $x(t)$ a solution in $\bar{G}$, for any given τ, and any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\begin{aligned} &V(x(\tau + \delta), \tau + \delta) - V(x(\tau), \tau) \\ &\quad \leqslant +\epsilon\delta + \int_t^{t+\delta} \left(\frac{\partial V(x(t), s)}{\partial x} \cdot f(x(t), s) + \frac{\partial V(x(t), s)}{\partial t} \right) ds \\ &\quad \leqslant -\delta(W(x) - \epsilon). \end{aligned} \tag{48}$$

If $\epsilon < W(x)$, it follows that V is less at $\tau + \delta$ than at τ.

If $E = \{x \in \bar{G}: W(x) = 0\}$ then it follows from standard Liapunov arguments (e.g., see [11]) that if a solution is bounded and stays in G, its positive limit set is in E.

Let M be the largest subset of E which is QIP relative to (47). Then the following standard type of Liapanuv theorem is an immediate consequence of Theorem 3.

COROLLARY 7. *If f satisfies the conditions of Theorem 2 and V is a Liapunov function for* (47) *on G, then all solutions that remain in G and are bounded on R^+ approach M as $t \to \infty$.*

The above result can be used to establish theorems on stability and instability (and unboundedness). We illustrate this by giving a sufficient condition in terms of a Liapunov function for uniform stability and for estimating regions of attraction. This is similar in form to LaSalle's Theorem 3 in [11].

THEOREM 4. *Let f satisfy the conditions of Theorem 2 and let V be a Liapunov function for* (47) *on G. In addition, assume that:*

(i) *G is a bounded open set that is positively invariant with respect to* (47);

(ii) *$\partial V/\partial x$ is bounded on $\bar{G} \times R^+$;*

(iii) *$M \subset G$ and $V(x, t) = c(t)$, a nondecreasing time-dependent constant on the boundary of M.*

If M is defined as the largest positive invariant subset of E, then M is a uniformly stable attractor and G is in its region of attraction (for all $t \geqslant 0$).

Proof. Assume that M is the largest subset of E which is quasi-invariant in the positive direction. The first step is to show that M is closed. Then, since M is a closed subset of the open set G, it will follow that for some $\gamma > 0$, $d(x, M) < \gamma$ implies that x is in G. We will show that the closure of an arbitrary positively quasi-invariant set is QIP.

Then it will follow, since M is maximal, that M must be closed. Let A be a QIP set containing $\{x_n\}_{n=1}^{\infty}$ and suppose $x_n \to x_0$ as $n \to \infty$. Then for each x_n there exists a sequence $\{t(n, i)\}_{n=1}^{\infty}$ such that $\phi(x_n, f_{t(n,i)}, t) \to \psi_n(t)$ as $i \to \infty$ and $t(n, i) \to \infty$ as $n \to \infty$. The convergence is uniform on compact subsets of R^+ and $\psi_n(t)$ is in A for all $t \in R$. Pick a diagonalization sequence $\{t_n\}$ such that $|\phi(x_n, f_{t_n}, t) - \psi_n(t)| < 1/n$ on $[-n, n]$. Using the compactness of $\mathscr{J}_f$, a subsequence (denoted $\{t_n\}$) can be found such that $f_{t_n} \to f^*$ in $\mathscr{J}_f$ as $n \to \infty$. Note that

$$|\phi(x_0, f^*, t) - \psi_n(t)| \leqslant |\phi(x_0, f^*, t) - \phi(x_0, f_{t_n}, t)| + |\phi(x_0, f_{t_n}, t) - \phi(x_n, f_{t_n}, t)| + |\phi(x_n, f_{t_n}, t) - \psi_n(t)|. \quad (49)$$

Given any finite interval I in R and any $\epsilon > 0$, clearly an N_1 can be found such that for $n > N_1$ the first and third terms on the right-hand side of (49) are each less that $\epsilon/3$. The Gronwall inequality can be used along with the Lipschitz bound for f to show for some $N_2 \geqslant N_1$ that the second term is less than $\epsilon/3$ for $n > N_2$. It then follows from (49) that $d(\phi(x_0, f^*, t), A) < \epsilon$ on I. However, ϵ and I are arbitrary; therefore, $\psi(x_0, f^*, t)$ is in the closure of A for all $t \geqslant 0$. Since x_0 is an arbitrary point in the closure of A it follows that the closure of A is QIP and M is closed.

It follows from Corollary 7 that M is an attractor with G in its region of attraction (for all $t_0 \geqslant 0$), hence it remains to show that M is uniformly stable; that is, given $\epsilon > 0$, there exists a $\delta(\epsilon)$, $\delta > 0$, such that, for any $t_0 \geqslant 0$, $d(\phi(x, f_{t_0}, t), M) < \epsilon$ for all x such that $d(x, M) < \delta$ and all $t \geqslant 0$. Assume therefore that M is not uniformly stable. Then there exists an $\epsilon > 0$ and sequences $\{x_n\}$, $\{t_n\}$, $\{t_n'\}$ such that t_n, $t_n' \geqslant 0$, $d(x_n, M) \to 0$ as $n \to \infty$ and $d(y_n, M) = \epsilon$ for all n where $y_n \equiv \phi(x_n, f_{t_n}, t_n')$ is the solution at $t_n + t_n'$ starting from x_n at t_n. Since G is bounded we may assume that these sequences are such that $x_n \to x_0 \in M$ and $y_n \to y_0$ and $d(y_0, M) = \epsilon$. From the assumption that $\partial V/\partial x$ is bounded on $G \times R^+$ and other assumptions it follows that V and c are bounded on $G \times R^+$ and $\lim_{n\to\infty} V(x_n, t_n) = \lim_{n\to\infty} c(t_n)$, if the limit exists. Also $\{f_t; t \in R^+\}$ is compact in $\mathscr{J}_f$ and we may assume for our sequences that $c(t_n) \to c_0$ and $f_{\tau_n} \to f^*$ as $n \to \infty$, where $\tau_n = t_n + t_n'$.

Suppose that $W(\phi(y_0, f^*, \tau)) = \alpha > 0$ for some $\tau \geqslant 0$. Then $W(\phi(y_n, f_{\tau_n}, \tau)) \geqslant \alpha/2$ for all sufficiently large n ($n \geqslant n_0$). Hence $\phi(y_n, f_{\tau_n}, \tau)$ is not in M for $n \geqslant n_0$, and there is a $w_n > 0$ such that $\phi(y_n, f_{\tau_n}, \tau + t)$ is not in M for $0 \leqslant t < w_n$ and $\phi(y_n, f_{\tau_n}, \tau + t) \to \partial M$ as $t \to w_n^-$ (w_n could be ∞). Then since the solution starting at x_n at time t_n goes to $\phi(y_n, f_{\tau_n}, \tau + t)$ at time $\tau_n + \tau + t$ and goes to ∂M as time $\to \tau_n + \tau + w_n^-$, it is easy to see that

$$V(\phi(y_n, f_{\tau_n}, \tau + t), \tau_n + \tau + t) \leqslant V(\phi(y_n, f_{\tau_n}, \tau), \tau_n + \tau) \leqslant V(x_n, t_n) \tag{50}$$

for $0 \leqslant t \leqslant w_n$ and $n \geqslant n_0$. From the boundedness of f on $G \times R^+$ and the assumptions on V and c one can then conclude that there is a $\delta > 0$ such that $c(t_n) + \delta\alpha \leqslant V(\phi(y_n, f_{\tau_n}, \tau), \tau_n + \tau) \leqslant V(x_n, t_n)$. Letting $n \to \infty$ leads to $c_0 + \delta\alpha \leqslant c_0$ and this implies $\alpha = 0$. Hence $\gamma = \{\phi(y_0, f^*, \tau); \tau \geqslant 0\}$ is in E and therefore in M, since the set γ is QIP. This contradicts the assumption that M is not uniformly stable. It is clear, since M is uniformly stable, that no solution can leave M and therefore M is positively invariant. Since M is the largest QIP set in E, it contains the largest positively invariant subset of E. Hence M is the largest positively invariant subset of E. This completes the proof.

In applications of this result it simplifies matters a great deal to know that M is the largest positively invariant subset of E, since largest positive invariance is usually easy to identify, whereas identifying positive quasi-invariance can be more difficult. The example given below illustrates this.

Note that if (a) V does not depend on t, (b) G is a bounded component of $\{x;\ V(x) < a\}$, and (c) M is a single point p, then (i)–(iii) are automatically satisfied and p is a uniformly stable attractor. The major gain, however, in studying stability is that $\dot{V}$ need not be negative definite.

We now give an example illustrating how this result can be applied. The global asymptotic stability of this nonlinear system, in slightly more general form, was studied in 1960 by Levin and Nohel [12]. They pointed out that classical Liapunov theory could not be applied and carried out a lengthy and complex analysis to obtain sufficient conditions. Using Theorem 4, we obtain sufficient conditions directly with no difficulty. Note also that Peng's result is not applicable and that in addition we are able to conclude uniform stability.

For the equation $\ddot{x} + h(x, y, t)\dot{x} + f(x) = 0$, or the equivalent system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -h(x, y, t)y - f(x), \end{aligned} \tag{51}$$

assume that:

(i) f is C^1, h, $\partial h/\partial x$, and $\partial h/\partial y$ exist and are continuous functions on $R^2 \times R^+$ and that these functions are all uniformly bounded for t in $[0, \infty)$ and (x, y) in any bounded set in R^2;

(ii) $h(x, y, t) \geqslant k(x, y) > 0$ for $y \neq 0$ and k is a continuous scalar function on R^2;

(iii) $xf(x) > 0$ for $x \neq 0$ and $S(x) = \int_0^x f(s)\, ds \to \infty$ as $|x| \to \infty$.

We shall show that the origin is a globally and uniformly stable attractor.

To see this, let $V(x, y) = \frac{1}{2}y^2 + S(x)$. Then $\dot{V}(x, y, t) = -h(x, y, t)\, y^2 \leqslant -k(x, y)\, y^2$. For any $c > 0$ take $G = \{(x, y);\ V(x, y) < c\}$. Then clearly G is open, bounded, and positively invariant, and E is the intersection of G and the x-axis. If $y = 0$ and $x \neq 0$, then $\dot{y} = -f(x) \neq 0$ and every such solution leaves the x-axis. Hence M is the origin and all the conditions of Theorem 4 are satisfied. Therefore, the origin is a uniformly stable attractor. Global stability follows since c was arbitrary and $V(x, y) \to \infty$ as $x^2 + y^2 \to \infty$.

The above example can be improved with some help from the perturbation results developed here.

Consider the following differential equations

$$\dot{x} = f'(x, t), \tag{52}$$

$$\dot{x} = f(x, t) + g(x, t), \tag{53}$$

$$\dot{x} = f(x, t) + g(x, t) + h(x, t), \tag{54}$$

and consider the assumptions listed below.

(H1) f satisfies the conditions of Theorem 2 and for $f' \in \mathscr{J}_f$ and $x(0) \in Z \subset R^n$, the positive limit sets for all bounded solutions to (52) lie in $E \subset R^n$.

(H2) g satisfies the conditions of Theorem 2 and $g_\tau \to 0$ in $\mathscr{I}$ as $\tau \to \infty$.

(H3) For some $t_0 > 0$, the set E is uniformly stable with respect to (53) on $[t_0, \infty)$.

(H4) There exists a Liapunov function V for (53) on $\bar{Z} \times R^+$.

(H5) If $x \notin E$, there exists a $t_1 > 0$ and an $\epsilon_x > 0$ such that $V(x, t) \geqslant V(x_0, \tau) + \epsilon_x$ for all x_0 on the boundary of E whenever $t \geqslant \tau \geqslant t_1$.

(H6) If $g \equiv h \equiv 0$, then $\dot{V} \leqslant -W(x) \leqslant 0$ at each x in Z; for t a.e.; W is continuous on Z; and $E_0 = \{x \in \bar{Z}\colon W(x) = 0\}$.

(H7) For any bounded set A in Z, $| \partial V/\partial x |$ is bounded on $A \times R^+$; h satisfies the conditions of Theorem 2; and $| h(x, t)| \leqslant \beta_A(t)$ for t a.e. at each x in A and

$$\int_0^\infty \beta_A(t)\, dt < \infty.$$

THEOREM 5.

(a) *If* (H1) *and* (H2) *hold and a solution to* (53) *remains in* Z *and is bounded, then there is some point* x_0 *in* E *such that* x_0 *is in the positive limit set for this solution.*

(b) *If* (H1), (H2), *and* (H3) *hold, then* E *contains the positive limit set for any bounded solution to* (53) *in* Z.

(c) *If* (H1), (H2), (H4), (H5), *and* (H7) *hold* (*or if* (H2), (H4), (H6), *and* (H7) *hold and* f *satisfies the conditions of Theorem* 2), *then the largest* QIP *subset of* E (*or* E_0) *with respect to* (52) *contains the positive limit set for any bounded solution to* (54) *in* Z.

Proof. (a) The positive limit set $L^+(f + g, x)$ is nonempty if (H1) holds and the solution is bounded. Suppose $x^* \in L^+(f + g, x)$. Then it follows from Corollary 5 that there exists a sequence $\{\tau_n\}$ such that $\tau_n \to \infty$ and $\phi(x, f + g, \tau_n + t) \to \phi(x^*, (f + g)^*, t)$ uniformly on compact sets in R as $n \to \infty$. However, from (H2), $(f + g)^* = f^* + g^* = f^*$ and $\phi(x^*, f^*, t) \to E$ as $t \to \infty$ in accordance with (H1). Therefore, given any $\epsilon > 0$, there exists an $x_\epsilon \in E$, a t_ϵ, and a τ_n such that $| \phi(x^*, f^*, t_\epsilon) - x_\epsilon | < \epsilon/2$, and $| \phi(x, f + g, \tau_n + t_\epsilon) - \phi(x^*, f^*, t_\epsilon)| < \epsilon/2$. It follows that $\phi(x, f + g, \tau_n + t_\epsilon)$ is within ϵ of x_ϵ in E. By choosing a sequence $\{1/k\}$ for ϵ, a sequence of t_k's are found such that $\phi(x, f + g, t_k)$ is within $1/k$ of E and $t_k \to \infty$ as $k \to \infty$.

Since the solution is bounded, a convergent subsequence exists and the limit point is in both E and the positive limit set for $\phi(x, f+g, t)$.

(b) This follows immediately from (a) and the definition of uniform stability.

(c) Note that from (H4),

$$\dot{V} = \frac{\partial V(x,t)}{\partial x}(f(x,t)+g(x,t)) + \frac{\partial V(x,t)}{\partial x} + \frac{\partial V(x,t)}{\partial x} h(x,t)$$
$$\leqslant -u(x) + \frac{\partial V(x,t)}{\partial x} h(x,t), \qquad \text{for} \quad u(x) \tag{55}$$

a continuous nonnegative function on Z.

It follows from Corollary 7 and Part (a) that some point x_0 in E (or E_0) is in the positive limit set (noting that (H7) implies $h_\tau \to 0$ in $\mathscr{I}$ as $\tau \to \infty$). Suppose x_1 is in the positive limit set and x_1 is not in E (or E_0). Then given any $\tau > 0$ and any $\delta > 0$ there exists $t_2 > t_1 > t$ such that $|\phi(x, f+g+h, t_1) - x_0| < \delta$; x_0 is on the boundary of E (or E_0); and $|\phi(x, f+g+h, t_2) - x_1| < \delta$. It follows from the definition of a Liapunov function that

$$V(\phi(x, f+g+h, t_2), t_2) - V(\phi(f+g+h, t_1), t_1)$$
$$= \int_{t_1}^{t_2} \dot{V}(\phi(x, f+g+h, t), t)\, dt \leqslant -\int_{t_1}^{t_2} u(\phi(x, f+g+h, t))\, dt$$
$$+ K\int_{t_1}^{t_2} \beta_A(t)\, dt \qquad \text{where} \quad K = \sup_{x\in A, t\in R^+} \left| \frac{\partial V(x_1 t)}{\partial x} \right|, \tag{56}$$

and A is a bounded set containing the solution in its interior.

$$V(\phi(x, f+g+h, t_2), t_2) \geqslant V(x_1, t_2) - K\,|\phi(x, f+g+h, t_2) - x_1| \tag{57}$$

and

$$V(\phi(x, f+g+h, t_1), t_1) \leqslant V(x_0, t_1) + K\,|\phi(x, f+g+h, t_1) - x_0|. \tag{58}$$

If (H5) holds, then $V(x_1, t_2) \geqslant V(x_0, t_1) + \epsilon_{x_1}$ for t_1 sufficiently large and $t_2 > t_1$. If t_1 and t_2 are chosen such that $|\phi(x, f+g+h, t_2) - x_1| \leqslant (\epsilon_{x_1}/4K)$ and $|\phi(x, f+g+h, t_1) - x_0| \leqslant (\epsilon_{x_1}/4K)$, and

$$\int_{t_1}^{\infty} \beta_A(t)\, dt \leqslant (\epsilon_{x_1}/4K),$$

noting that $\dot{V} \leqslant 0$, combining (56), (57), and (58) gives

$$\begin{aligned} V(x_1, t_2) - V(x_0, t_1) &\\ \leqslant V(\phi(x, f+g+h, t_2), t_2) &\\ - V(\phi(x, f+g+h, t_1), t_1) + K \mid \phi(x_1, f+g+h, t_2) - x_1 \mid &\\ + K \mid \phi(x, f+g+h, t_1) - x_0 \mid \leqslant (3\epsilon_{x_1}/4) & \qquad (59) \end{aligned}$$

which contradicts (H5). It follows that the positive limit set is in E.

Now suppose (H6) holds. Let

$$V_1 = V(x, t) + \int_t^\infty K\beta_A(s)\, ds.$$

Then $\dot{V}_1(x, t) = \dot{V}(x, t) + \partial V(x, t)/\partial x \cdot h(x, t) - K\beta_A(t) \leqslant 0$ on A since V is a Liapunov function for (53). It follows that V_1 is a Liapunov function for (54) on A.

Let $W(x_1) = a > 0$. Then there exists a neighborhood of radius δ around x_1 such that $W(x) > a/2$ and such that $\delta < \max\{2M_A/9Q_A, a/12KQ_A\}$ where $Q_A = \sup_{x\in(A-x_1), t\in R^+} (| g(x, t) - g(x_1, t)|/| x - x_1 |)$, a Lipschitz bound for g, and M_A is a bound for f on A. Since x_1 is in the positive limit set of the solution, $\phi(t)$, there exists a sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$ and $| \phi(t_n) - x_1 | < \delta/2$;

$$\int_{t_n}^\infty \beta_A(t)\, dt \leqslant \max\{\delta/12, \delta a/32KM_A\};$$

and

$$\left| \int_{t_n}^{t_n+\delta/4M_A} g(x, s)\, ds \right| \leqslant \max\{\delta/12, a/8K\}.$$

The last inequality is assured since $g_\tau \to 0$ in $\mathscr{I}$ as $\tau \to \infty$. Note that

$$\begin{aligned} &\left| \int_{t_n}^{t_n+\delta/4M_A} \dot{\phi}(t)\, dt \right| \\ &\quad \leqslant \left| \int_{t_n}^{t_n+\delta/4M_A} \{f(\phi(t), t) + g(\phi(t), t) + h(\phi(t), t)\} \right| \\ &\quad \leqslant \delta/4 + \delta/12 + \int_{t_n}^{t_n+\delta/4M_A} | g(\phi(t_n), t) - g(\phi(t), t)|\, dt \\ &\qquad + \int_{t_n}^\infty \beta_A(t)\, dt < \frac{5\delta}{12} + \frac{\delta}{4M_A} \frac{(3\,\delta Q_A)}{2} \leqslant \frac{\delta}{2} \text{ (from the bound on } \delta). \end{aligned}$$

These inequalities are valid as long as $\phi(t)$ stays within δ of x. It follows that $\phi(t)$ stays within the δ radius of x_1 on the interval $[t_n, t_n + \delta/4M_A]$.

$$V_1(\phi(x, f+g+h, t_n+\delta/4M_A), t_n+\delta/4M_A) - V_1(\phi(x, f+g+h, t_n), t_n)$$

$$\leqslant \int_{t_n}^{t_n+\delta/4M_A} \left(\frac{\partial V(\phi(t), t)}{\partial x} \cdot f(\phi(t), t) + \frac{\partial V(\phi(t), t)}{\partial t}\right) dt$$

$$+ \int_{t_n}^{t_n+\delta/4M_A} \frac{\partial V(\phi(t), t)}{\partial x} \cdot g(\phi(t), t)\, dt + \int_{t_n}^{\infty} \left| \frac{\partial V(\phi(t), t)}{\partial x} \right| \beta_A(t)\, dt$$

$$\leqslant -\frac{a}{2}(\delta/4M_A) + \delta/4M_A(K)(a/8K)$$

$$+ \int_{t_n}^{t_n+\delta/4M_A} K\, |g(\phi(t_n), t) - g(\phi(t), t)|\, dt$$

$$+ \delta a/32M_A \leqslant -\delta a/16M_A + K\,\delta/4M_A(3/2\,\delta Q_A) \leqslant -\delta a/32M_A\,. \tag{61}$$

V_1 is a monotonic decreasing function of time along the solution. However, it passes within $\delta/2$ of x_1 infinitely often, each time decreasing by at least $a\delta/32M_A$. Therefore, $V_1 \to -\infty$ as $t \to \infty$, but this contradicts the assumption that V is a Liapunov function. It follows that the positive limit set lies in E_0.

Corollary 7 implies that the positive limit set is in the largest QIP subset of E_0 relative to $f+g+h$. However, since (H7) implies that $h^* = 0$, hence $(f+g+h)^* = f^*$ and the largest QIP subset of E_0 relative to $f+h+h$ is the same as the subset relative to f. The same argument applies for the set E and the proof is complete.

The usual Liapunov type methods can be used to establish boundedness of solutions.

Sufficient conditions for g_t to approach 0 in $\mathscr{F}$ as $t \to \infty$ are (i) $g(x, t) \to 0$ uniformly on compact sets in R^n as $t \to \infty$, or (ii) given any compact set A in W and $\epsilon > 0$, there is a $T = T(\epsilon, A)$ such that

$$\int_T^{\infty} |g(x, s)|\, ds < \epsilon$$

for all x in A, or (iii) there is a $T(\epsilon, A)$ such that for

$$t > T, 0 \leqslant s \leqslant 1, \left| \int_t^{t+s} g(x, \lambda)\, d\lambda \right| < \epsilon$$

for all x in A. It can be shown that (iii) implies that $g_t \to 0$ in $\mathscr{I}$ as $t \to \infty$ since

$$\int_0^s g(x, t + \lambda)\, d\lambda \to 0$$

uniformly on compact sets in (x, s) as $t \to \infty$.

The above result follows for f autonomous from a result of Strauss and Yorke [15]. They obtain a condition on g which implies (iii) and although they assume g is continuous in (x, t), their proof holds for g satisfying the conditions of Theorem 2.

Using these results, the stability property for the example (50) can be extended along lines like those considered by Onuchic [16].

$$\begin{gathered} \dot{x} = y, \\ \dot{y} + H(x, y, t)y + F(x) + G(x, y, t) + p(x, y, t) + q(x, y, t) = 0, \end{gathered} \tag{62}$$

and the following set of assumptions with respect to (62).

(i) The function $H(x, y, t)$ satisfies the conditions of Theorem 2 and for each (x, y) in R^2, $H(x, y, t) \geqslant k(x, y) \geqslant 0$ for t a.e. and $k(x, y)$ is a continuous scalar function on R^2.

(ii) For every Jordan curve γ in R^2 containing the origin in its interior, there exists at least one point $(x, y) \in \gamma$ such that $yk(x, y) \neq 0$.

(iii) $F(x)$ is locally L_1, $xF(x) > 0$ for all $x \neq 0$ and

$$\left| \int_0^x F(s)\, ds \right| \to \infty \qquad \text{as} \quad |x| \to \infty.$$

(iv) G satisfies the conditions of Theorem 2 and for each (x, y) in R^2, $yG(x, y, t) \geqslant 0$ for t a.e.

(v) p satisfies the conditions of Theorem 2; $p_\tau \to 0$ in $\mathscr{I}$ as $\tau \to \infty$; and at each x, y in R^2, $y^2H(x, y, t) + y(G(x, y, t) + p(x, y, t)) \geqslant 0$ for t a.e.

(vi) q satisfies the conditions of Theorem 2; on any bounded set A, $q(x, y, t) \leqslant \beta_A(t)$ where

$$\int_0^\infty \beta_A(s)\, ds < \infty;$$

and there exists a Liapunov function V_1 for (62), excluding q such that for any (x_0, y_0, t_0) in $R^2 \times R^+$, there exists an $r > 0$ such that

$$\operatorname*{Sup}_{\substack{(x,y)\in S_r \\ t>t_0}} \left| \frac{\partial V_1(x, y, t)}{\partial y} \right| \int_{t_0}^{\infty} \beta_{S_r}(s)\, ds < \inf_{\substack{\{x^2+y^2=r^2\} \\ t>t_0}}$$

$$(V_1(x, y, t) - V_1(x_0, y_0, t_0)) \text{ where } S_r \equiv \{(x, y)\colon x^2 + y^2 \leqslant r^2\}.$$

COROLLARY 8. *Under conditions* (i) *through* (vi) *above, for each solution to* (62), $x(t) \to 0$ *and* $y(t) \to 0$ *as* $t \to \infty$.

Proof. Let

$$V(x, y, t) = \left[y^2 + 2\int_0^x F(s)\, ds\right]^{1/2}.$$

Then

$$\begin{aligned} \dot V(x, y, t) &= \frac{-y^2 H(x, y, t) - yG(x, y, t) - yp(x, y, t) - yq(x, y, t)}{[y^2 + 2\int_0^x F(s)\, ds]^{1/2}} \\ &\leqslant | q(x, y, t)|. \end{aligned} \tag{63}$$

Using $\begin{pmatrix} 0 \\ -p \end{pmatrix}$ as g, $\begin{pmatrix} 0 \\ -g \end{pmatrix}$ as h, $\begin{pmatrix} y \\ -H(x, y, t)\, y - F(x) - G(x, y, t) \end{pmatrix}$ as f in Theorem 5, (H2), (H4), (H6) and (H7) apply with $W(x, y) = -y^2 k(x, y)$ and $E_0 = x$-axis plus $\{(x, y)\colon k(x, y) = 0\}$. $\partial V/\partial x$ is bounded on any bounded set in R^2 since V is C_1 and does not depend on t. Therefore, all bounded solutions approach the largest subset of E_0 which is QIP with respect to (62) when $p = q \equiv 0$. However, since V decreases, the positive limit set must consist of points on which V is constant. The set of points on which V is cosntant is either the origin or a Jordan curve containing the origin in the interior. If it is a Jordan curve, it is easily shown from the conditions on H and G that a limiting solution $(\phi(x^*, F^*, t))$ must advance around the curve in finite time. Therefore, there is a point on the curve where $W(x, y) \neq 0$, a contradiction. It follows that the positive limit set is the origin or $x(t) \to 0$ and $\dot x(t) = y(t) \to 0$ as $t \to \infty$.

The boundedness of the solution follows from (vi) by noting that

$$\dot V_1 = \frac{\partial V_1}{\partial x} y + \frac{\partial V_1}{\partial y}(-yH - F - G - p - q) + \frac{\partial V_1}{\partial t} \leqslant \left| \frac{\partial V_1}{\partial y} q \right|,$$

and

$$\int_{t_0}^{t} \dot V_1(x(s), y(s), s)\, ds \leqslant \sup_{\substack{(x,y)\in S_r \\ t>t_0}} \left| \frac{\partial V_1(x, y, t)}{\partial y} \right| \int_{t_0}^{t} \beta_{S_r}(s)\, ds \tag{64}$$

for $(x(s), y(s))$ in S_r on $[t_0, t]$. It follows from (vi) that $V_1(x(t), y(t), t) < V_1(x, y, t)$ for all (x, y) satisfying $x^2 + y^2 = r^2$ and therefore, the solution can never cross the boundary of S_r and is, therefore, bounded.

8. Generalizations

The results in this paper can be applied to quasi-invariance in the negative direction or quasi-invariance (both directions) in a direct way for appropriate functions defined on $W \times R$.

A topology similar to $\mathscr{I}$ may be applicable with certain limitations to Volterra integral equations as considered by Miller and Sell [3–5] or to functional differential equations to obtain more useful conditions for compact motion. The primary difficulty is that bounded sets in the domain of the function are no longer compact.

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